परमाणु ऊर्जा शिक्षण संस्था

(परमाणु ऊर्जा विभाग का स्वायत्त निकाय, भारत सरकार)

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CLASS XII (MATHEMATICS)

CHAPTER – 09

DIFFERENTIAL EQUATIONS

MODULE – 2/3

METHODS OF SOLVING FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS

- (1) **Differential equations with variables separable:**
- If a first order-first degree differential equation

$$\frac{\mathbf{dy}}{\mathbf{dx}} = f(x, y) \text{ ----- } (1)$$

can be expressed in the form $\frac{\mathbf{dy}}{\mathbf{dx}} = g(x).\ h(y)$,

then the differential equation (1) is said to be of variable separable type.

- **How to solve:** $\frac{dy}{h(y)} = g(x)dx$

$$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)dx, \quad h(y) \neq 0$$

Solution of the differential equation: H(y) = G(x) + C

WORKING STEPS TO SOLVE $\frac{dy}{dx} = g(x).\, h(y)$

- First identify the functions g(x) and h(y)
- Bring the expression containing x, i.e. g(x) on any one side of equal sign with dx in the numerator & the expression containing y, i.e. h(y) on the other side of equal sign with dy in the numerator. Always keep dx & dy in the numerators.
- Apply integration on both the sides and integrate using proper methods. Add arbitrary constant C only on one side. This will give you the general solution of the given differential equation.
- If some initial conditions are given , then apply them to find the value of C & put the value of C in the general solution to get particular solution.

EXAMPLES:

(1) Find the general solution of the differential equation

$$\frac{dy}{dx} = 1 - x + y - xy$$

- Ans: $\frac{dy}{dx} = 1 - x + y - xy$

$$= (1 - x) + y\,(1 - x)$$

$$= (1 - x)\,(1 + y)$$

Now separate the variables

$$\frac{dy}{1+y} = (1-x)dx$$

Integrate both the sides

$$\int \frac{dy}{1+y} = \int (1-x)dx$$

$$\log|1+y| = x - \frac{x^2}{2} + C.$$

(2) Find the general solution of the differential equation

$$e^x \tan y\, dx + (1 - e^x)\, Sec^2 y\, d y = 0$$

- Ans: $e^x \tan y\, dx + (1 - e^x)\, Sec^2 y\, d y = 0$

$$.(1 - e^x)\, Sec^2 y\, d y = - e^x \tan y\, dx$$

$$\frac{Sec^2 y\, d y}{\tan y} = -\frac{e^x\, dx}{(1 - e^x)} \Rightarrow \int \frac{Sec^2 y\, d y}{\tan y} = -\int \frac{e^x\, dx}{(1 - e^x)}$$

Put $\tan y = t$ Put $1 - e^x = u$

$Sec^2 y\, d y = dt$ $- e^x\, dx = du$

$$\int \frac{dt}{t} = \int \frac{du}{u}$$

$$\Rightarrow \log |t| = \log |u| + \log C$$

$$\log |\tan y| = \log |C(1 - e^x)|$$

$$|\tan y| = |C(1 - e^x)|$$

(3) Find the particular solution of the differential equation $\frac{dy}{dx}$ = y tan x , given that $y = 1$, when $x = 0$.

- Ans: $\frac{\mathrm{dy}}{\mathrm{dx}}$ = y tan x

$$\frac{dy}{y} = tanx\,dx \Rightarrow \int \frac{dy}{y} = \int tanx\,dx$$

$\Rightarrow$ log $|y|$ = log $|Sec\ x|$ + log C

log $|y|$ = log C$|Sec\ x|$

$|y|$ = C$|Sec\ x|$ --- (i)

Put y = 1 & x = 0

1 = C . 1 $\Rightarrow$ C = 1

from (i) y = Sec x which is a particular solution.

(4) Find the equation of a curve passing through the point (0, –2) given that at any point (*x*, *y*) on the curve, the product of the slope of its tangent and *y* coordinate of the point is equal to the *x* coordinate of the point.

- Ans: Let the equation of the curve be y = f(x)

$\therefore$ Slope of the tangent at a point P(x, y) to the curve = $\frac{dy}{dx}$

So by the given conditions , $\frac{dy}{dx} . y = x$

which is a differential equation in the form “variable separable”

$$\therefore \; y\, d\, y = x\, dx \Rightarrow \int y\, dy = \int x\, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C \text{ ---- (i)}$$

Now this curve passes through the point (0, –2)

from (i) $\frac{4}{2} = 0 + C \quad \Rightarrow \quad C = 2$

from (i) $\frac{y^2}{2} = \frac{x^2}{2} + 2$

$$y^2 = x^2 + 4 \Rightarrow y^2 - x^2 = 4$$

which is required equation of the curve.

(2) HOMOGENEOUS DIFFERENTIAL EQUATIONS:

- **Homogeneous function:**

A function $F(x, y)$ is said to be homogeneous function of degree n if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

e.g. (i) $F(x, y) = y^3 + 2x^2y$

$\Rightarrow \quad F(\lambda x, \lambda y) = (\lambda y)^3 + 2(\lambda x)^2 . \lambda y$

$= \lambda^3[y^3 + 2x^2y]$

$= \lambda^3 F(x, y)$

$\therefore F(x, y) = y^3 + 2x^2y$ *is a* homogeneous function of degree *3*

(ii) $F(x, y) = \text{Cos}\left(\frac{y}{x}\right)$ *is a* homogeneous function of degree 0

(iii) $F(x, y) = (\text{Sin } x - \tan y)$ *is not a* homogeneous function

- **Homogeneous Differential Equation:-**

A differential equation of the form $\frac{dy}{dx}$ = F (x, y) is said to be homogenous differential equation if F(x, y) is a homogenous function of degree zero.

e.g. (i) $$\frac{dy}{dx} = \frac{(x + 3y)}{(x - y)} = F(x, y)$$

Replacing *x* by λ *x* and *y* by λ *y*, we get

$\frac{dy}{dx} = \frac{(\lambda x + 3\lambda y)}{(\lambda x - \lambda y)} = \lambda^0 \frac{(x + 3y)}{(x - y)} = \lambda^0$ F (x, y) which is a homogenous function of degree zero.

$\frac{dy}{dx} = \frac{(x + 3y)}{(x - y)}$ is a homogenous differential equation.

METHOD OF SOLVING A HOMOGENEOUS DIFFERENTIAL EQUATION:-

Let $\frac{dy}{dx} = F\,(x, y) = g\left(\frac{y}{x}\right)$ be a homogeneous differential equation.

Make the substitution $y = v.\, x$

Diff. w. r. to x, we get

$$\frac{dy}{dx} = v.\,1 + x.\,\frac{dv}{dx}$$

Substituting the value of $\frac{dy}{dx}$ in the given differential equation, we get

$$v.\,1 + x.\,\frac{dv}{dx} = g\,(v)$$

$$x.\,\frac{dv}{dx} = g\,(v) - v$$

$$\frac{dv}{g\,(v) - v} = \frac{dx}{x}$$

Integrating both the sides, we get

$$\int \frac{dv}{g\,(v) - v} = \int \frac{dx}{x}$$

Which gives the general solution of the given differential equation when v is replaced by $\frac{y}{x}$.

EXAMPLES:

- (i) Solve the differential equation $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

Ans: $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$ --------(i)

Clearly (i) is a homogeneous differential equation. Put y = v x

Diff. w. r. to x, we get, $\frac{dy}{dx} = v.\ 1 + x.\ \frac{dv}{dx}$

From (i) $v.\ 1 + x.\ \frac{dv}{dx} = \frac{x^2+x.vx+(vx)^2}{x^2}$

$$v.\ 1 + x.\ \frac{dv}{dx} = 1 + v + v^2$$

$$\Rightarrow \quad x.\ \frac{dv}{dx} = 1 + v^2 \quad \Rightarrow \frac{dv}{1+v^2} = \frac{dx}{x} \quad \Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$tan^{-1}v = log|x| + C \quad \Rightarrow tan^{-1}\frac{y}{x} = log|x| + C$$

(ii). Solve $x \operatorname{Cos}\left(\frac{y}{x}\right) . \frac{dy}{dx} = y \operatorname{Cos}\left(\frac{y}{x}\right) + x$

- Ans: $x \operatorname{Cos}\left(\frac{y}{x}\right) . \frac{dy}{dx} = y \operatorname{Cos}\left(\frac{y}{x}\right) + x$

$\frac{dy}{dx} = \frac{\frac{y}{x} \operatorname{Cos}\left(\frac{y}{x}\right)+1}{\operatorname{Cos}\left(\frac{y}{x}\right)}$ -------- (i) is a homogeneous DE. Put y = v x

Diff. w. r. to x, we get $\frac{dy}{dx} = v.\, 1 + x.\, \frac{dv}{dx}$

From (i)

$$v.\, 1 + x.\, \frac{dv}{dx} = \frac{v \operatorname{Cos} v+1}{\operatorname{Cos} v} \Rightarrow x.\, \frac{dv}{dx} = \frac{v \operatorname{Cos} v+1}{\operatorname{Cos} v} - v = \frac{v \operatorname{Cos} v+1-v \operatorname{Cos} v}{\operatorname{Cos} v}$$

$$x.\, \frac{dv}{dx} = \frac{1}{\operatorname{Cos} v} \Rightarrow \operatorname{Cos} v\, dv = \frac{dx}{x} \Rightarrow \int \operatorname{Cos} v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \operatorname{Sin} v = \log|x| + C \Rightarrow \operatorname{Sin}\left(\frac{y}{x}\right) = \log|x| + C$$

(iii) Solve $2ye^{\frac{x}{y}}.dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$

- Ans: $2ye^{\frac{x}{y}}.dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$

$$2ye^{\frac{x}{y}}.dx = -\left(y - 2xe^{\frac{x}{y}}\right)dy$$

$\frac{dx}{dy} = \frac{-\left(y-2xe^{\frac{x}{y}}\right)}{2ye^{\frac{x}{y}}} = \frac{\left(2xe^{\frac{x}{y}}-y\right)}{2ye^{\frac{x}{y}}} = \frac{\left(2\frac{x}{y}e^{\frac{x}{y}}-1\right)}{2e^{\frac{x}{y}}}$ ---(i) which is a homogeneous DE.

Put x = v y, Diff. w. r. to y, we get, $\frac{dx}{dy} = v.\,1 + y.\,\frac{dv}{dy}$

from (i) $v.\,1 + y.\,\frac{dv}{dy} = \frac{2ve^{v}-1}{2e^{v}} \Rightarrow$ $y.\,\frac{dv}{dy} = \frac{2ve^{v}-1}{2e^{v}} - v = \frac{2ve^{v}-1-2ve^{v}}{2e^{v}}$

$y.\,\frac{dv}{dy} = -\frac{1}{2e^{v}}$ $\Rightarrow 2e^{v}\,dv = -\frac{dy}{y} \Rightarrow$ $2\int e^{v}\,dv = -\int\frac{dy}{y}$

$2e^{v} = -\log|y| + \log C$ $\Rightarrow$ $2ye^{\frac{x}{y}} = \log\left|\frac{C}{y}\right|,\ y \neq 0$