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CLASS XII (MATHEMATICS)

CHAPTER – 09

DIFFERENTIAL EQUATIONS

MODULE – 3/3

METHODS OF SOLVING FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS

- **(3) Linear differential equations:**
- A differential equation of the from

$$\frac{\mathbf{dy}}{\mathbf{dx}} + \mathrm{P}\,\mathbf{y} = \mathrm{Q}$$

where, P and Q are constants or functions of *x* only, is known as a first order linear differential equation.

- Examples: (i) $\frac{\mathbf{dy}}{\mathbf{dx}} + 2\,\mathbf{y} = e^{5x}$

(ii) $\frac{\mathbf{dy}}{\mathbf{dx}} + \frac{\mathbf{1}}{\boldsymbol{x}}\,\mathbf{y} = \mathrm{Cos}\ x$

(iii) $\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{y\ sec\ x} = 7$

How to solve the first order linear differential equation of the type $\frac{dy}{dx} + P\,y = Q.$

- Given linear differential equation is

$$\frac{dy}{dx} + P\,\mathbf{y} = Q \text{ ----------(i)}$$

Multiply both sides of the equation (i) by a function of x say $g\,(x)$,

$$g\,(x).\frac{dy}{dx} + P.\,g\,(x).\,\mathbf{y} = Q.\,g\,(x) \text{ ----------(ii)}$$

Choose $g\,(x)$ in such a way that R.H.S. of (ii) i.e. $Q.\,g\,(x)$ becomes a derivative of $y\,.\,g\,(x)$.

$$\Rightarrow g\,(x).\frac{dy}{dx} + P.\,g\,(x).\,\mathbf{y} = \frac{d}{dx}(y\,.\,g\,(x))$$

Contd.

- $$\Rightarrow g\,(x).\,\frac{\mathbf{dy}}{\mathbf{dx}} + \text{P}.\,g\,(x).\,\mathbf{y} = g\,(x).\,\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{y}\,.\,\boldsymbol{g'(x)}$$

$$\Rightarrow \qquad \text{P}.\,g\,(x) = \boldsymbol{g'(x)}$$

$$\Rightarrow \qquad \text{P} = \frac{\boldsymbol{g'(x)}}{g\,(x)}$$

Integrating both sides with respect to *x*, we get

$$\int P\,dx = \int \frac{\boldsymbol{g'(x)}}{g\,(x)}\,dx$$

$$= \log\,(\text{g}(\text{x}))$$

$$\text{g}(\text{x}) = e^{\int P\,dx}$$

- On multiplying the equation (i) by g(x) = $e^{\int P\,dx}$, the L.H.S. becomes the derivative of some function of *x* and *y*.

Contd.

- This function g(x) = $e^{\int P\,dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of *g* (*x*) in equation (ii), we get

$$e^{\int P\,dx} . \frac{\mathbf{dy}}{\mathbf{dx}} + \mathrm{P}.\, e^{\int P\,dx} . \mathbf{y} = \mathrm{Q}.\, e^{\int P\,dx}$$

$$\frac{\mathbf{d}}{\mathbf{dx}}\left(\boldsymbol{y}.\, e^{\int P\,dx}\right) = \mathrm{Q}.\, e^{\int P\,dx}$$

Integrating both sides with respect to *x*, we get

$$\boldsymbol{y}.\, e^{\int P\,dx} = \int\left(\mathrm{Q}.\, e^{\int P\,dx}\right) dx + \mathrm{C}$$

Which is a general solution of the DE (i)

∴ We can use Solution formula as: **y . (I.F.) = $\int \boldsymbol{Q}\ (\boldsymbol{I}.\boldsymbol{F}.)\ \boldsymbol{dx}$ + C**

Working steps to solve first order linear differential equation:

- (i) Write the given differential equation in the form

$$\frac{dy}{dx} + P\,y = Q$$

- (ii) Find the Integrating Factor,

$$\text{I.F} = e^{\int P dx}$$

- (iii) Write the solution formula of the given differential equation as:

$$y.\,(\text{I.F}) = \int Q.(I.F)dx$$

- which gives the general solution after integration on right hand side.

EXAMPLES:

(1) Find the general solution of the differential equation

$$\frac{\mathrm{dy}}{\mathrm{dx}} + 4\mathrm{y} = e^{-3x}$$

- Ans: Given DE is $\frac{\mathrm{dy}}{\mathrm{dx}} + 4\mathrm{y} = e^{-3x}$

  Clearly this DE is in linear form ($\frac{\mathrm{dy}}{\mathrm{dx}}$ + P y = Q)

  P = 4, Q = e^{-3x}

  I. F. = $e^{\int pdx}$ = $e^{\int 4dx}$ = e^{4x}

  Solution formula, y. (I.F) = $\int Q.(I.F)dx$

  y. e^{4x} = $\int e^{-3x}.e^{4x}dx$

  = $\int e^{x}dx$

  = e^{x} + C

  y. e^{4x} = e^{x} + C

  which is a general solution of the given DE.

(2) Find the general solution of the differential equation

$$Cos^2x.\frac{dy}{dx} + y = \tan x, \quad \left(0 \le x < \frac{\pi}{2}\right)$$

- Ans: Given DE can be written as $\frac{dy}{dx} + y\,Sec^2x = Sec^2x.\tan x,$

Which is a linear DE of the form $\frac{dy}{dx}$ + P y = Q, P = Sec^2x , Q = $Sec^2x.\tan x,$

I. F. = $e^{\int p dx}$ = $e^{\int Sec^2x\, dx}$ = e^{tanx}

Solution formula, y. (I.F) = $\int Q.(I.F)dx$

y. e^{tanx} = $\int Sec^2x.\tan x \;.\, e^{tanx}\, dx$

Put tan x = t $\Rightarrow$ $Sec^2x\, dx = dt,$ y. e^{tanx} = $\int t.e^t dt$

Integrating by part, y. e^{tanx} = $t.e^t - \int e^t dt$ = $t.e^t - e^t$ + C

y. e^{tanx} = $\tan x\, .\, e^{\tan x} - e^{\tan x}$ + C or y = $(\tan x - 1)$ + C e^{-tanx}

(3) Find the particular solution of the differential equation

$\frac{dy}{dx}$ - 3 y Cot x= Sin 2x , given that y = 2, when $x = \frac{\pi}{2}$.

- Ans: Given DE is in linear form($\frac{dy}{dx}$ + P y = Q) ,

P = - 3 Cot x, Q = Sin 2x

I. F. = $e^{\int pdx}$ = $e^{\int -3 \text{ Cot x } dx}$ = $e^{-3 \log Sinx}$ = $(Sinx)^{-3}$

Solution formula, y. (I.F) = $\int Q.(I.F)dx$

y.$(Sinx)^{-3}$= $\int Sin\ 2x\ .\ (Sinx)^{-3} dx$

$= \int \frac{2\ Sinx\ Cos\ x}{Sin^3 x} dx = 2 \int Cosec\ x. Cot\ x\ dx$

y.$(Sinx)^{-3}$ = - 2 Cosec x + C ------ (i)

Put y = 2 & x = $\frac{\pi}{2}$ in (i) , 2 = -2 + C , C = 4

From (i) y.$(Sinx)^{-3}$ = - 2 Cosec x + 4,

y = - 2 $Sin^2 x$ + 4 $Sin^3 x$ which is particular sol.

(4) Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (*x*, *y*) is equal to the sum of the coordinates of the point.

- Ans: The slope of the tangent to the curve at any point (*x*, *y*) = $\frac{dy}{dx}$

According to question $\frac{dy}{dx}$ = x + y $\Rightarrow$ $\frac{dy}{dx}$ - y = x

Which is a linear DE of the form $\frac{dy}{dx}$ + P y = Q, P = - 1 , Q = x

I. F. = $e^{\int p dx}$ = $e^{-\int dx}$ = e^{-x}

Solution formula, y. (I.F) = $\int Q.(I.F)dx$

y. e^{-x} = $\int x\, e^{-x} dx$

= - $x\, e^{-x} + \int e^{-x} dx$

= - $x\, e^{-x}$ - e^{-x} + C ------ (i)

This curve passes through origin, put x =0, y = 0 in (i), C = 1

y. e^{-x} = - $x\, e^{-x}$ - e^{-x} + 1 $\Rightarrow$ x + y + 1 = e^{x}

which is required equation of curve