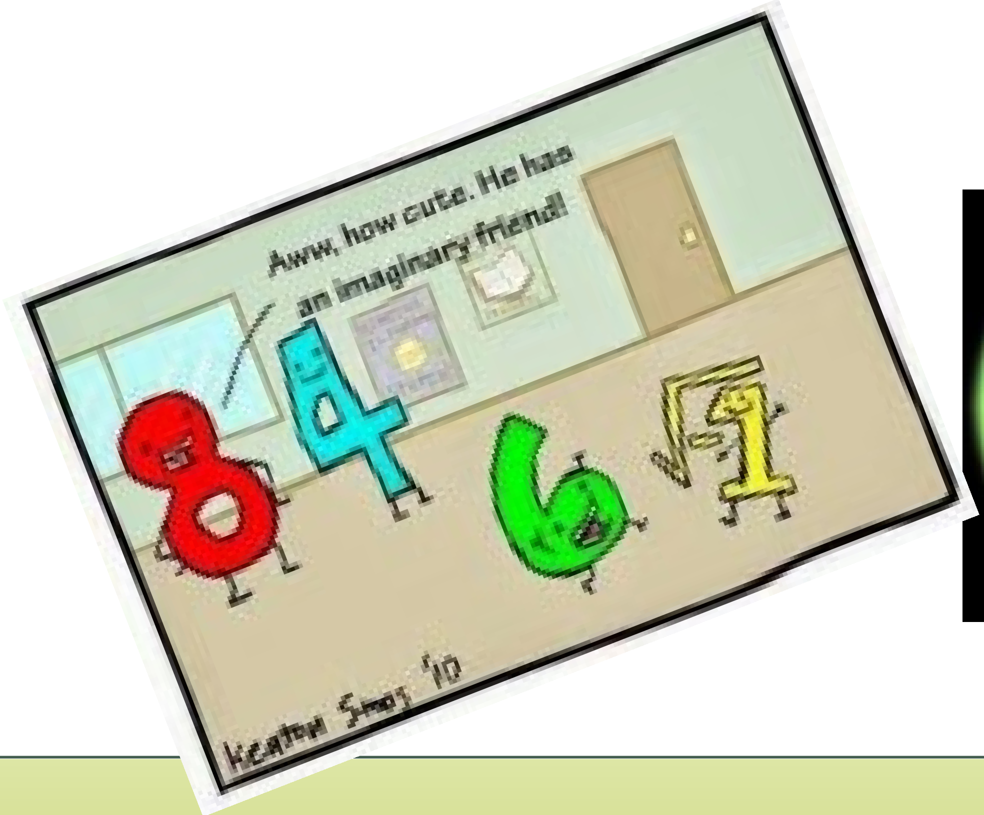
परमाणु ऊर्जा शिक्षण संस्था
(परमाणु ऊर्जा विभाग का स्वायत्त निकाय, भारत सरकार)
ATOMIC ENERGY EDUCATION SOCIETY
(An autonomous body under Department of Atomic Energy, Govt. of India)

5. Complex Numbers & Quadratic Equations



Module I

e-content

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Introduction :

We have studied linear equations in one and two variables and quadratic equations in one variable.

We have seen that the equation $x^2 + 1 = 0$ has no real solution,

as $x^2 + 1 = 0$

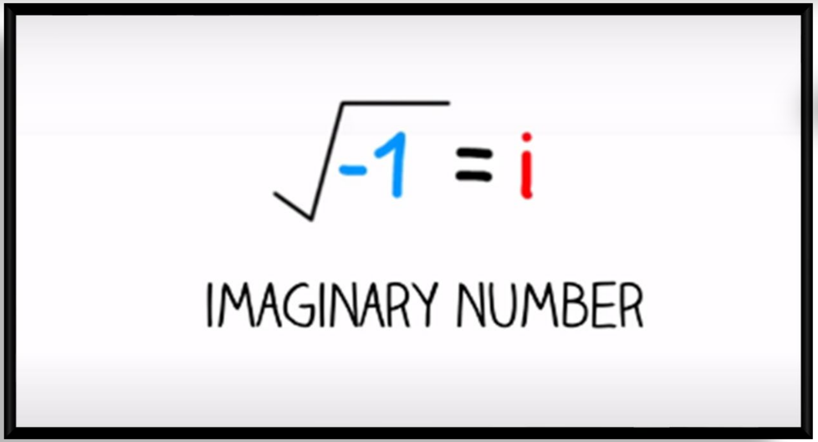
$x^2 = -1$

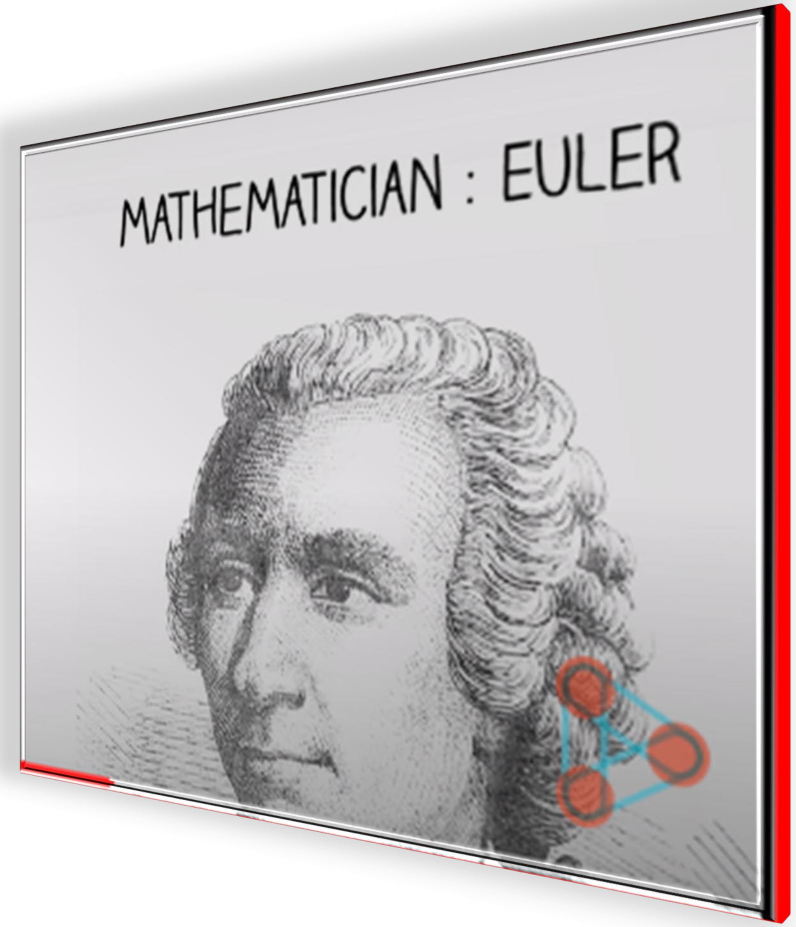
and square of every real number is non-negative. So, we need to extend the **real number system** to a larger system so that we can find the solution of the equation $x^2 = -1$.

In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

So, we will take $\sqrt{-1} = i$ as Mathematician Euler who gave this symbol.

Mathematicians – Carden & Euler





Definition of Complex Numbers

If a and b are real numbers and i is the imaginary unit, then $a + bi$ is called a complex number. And denoted by z, if we take two or more complex numbers then we will denote them as $z_1, z_2, z_3, \ldots\ldots\ldots etc.$

- The lowercase iota symbol is used to write the imaginary unit, but more often Roman i is used.

PROBLEM 1

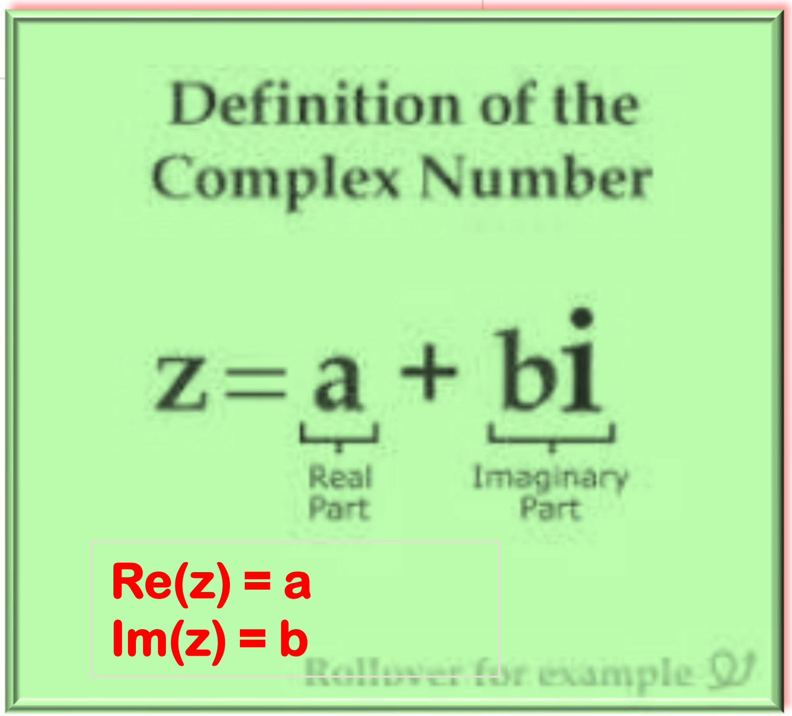
What is the real part of 13.2i + 113? 113

PROBLEM 2

What is the imaginary part of 21 - 14i? -14

PROBLEM 3

What is the real part of 17i? 0

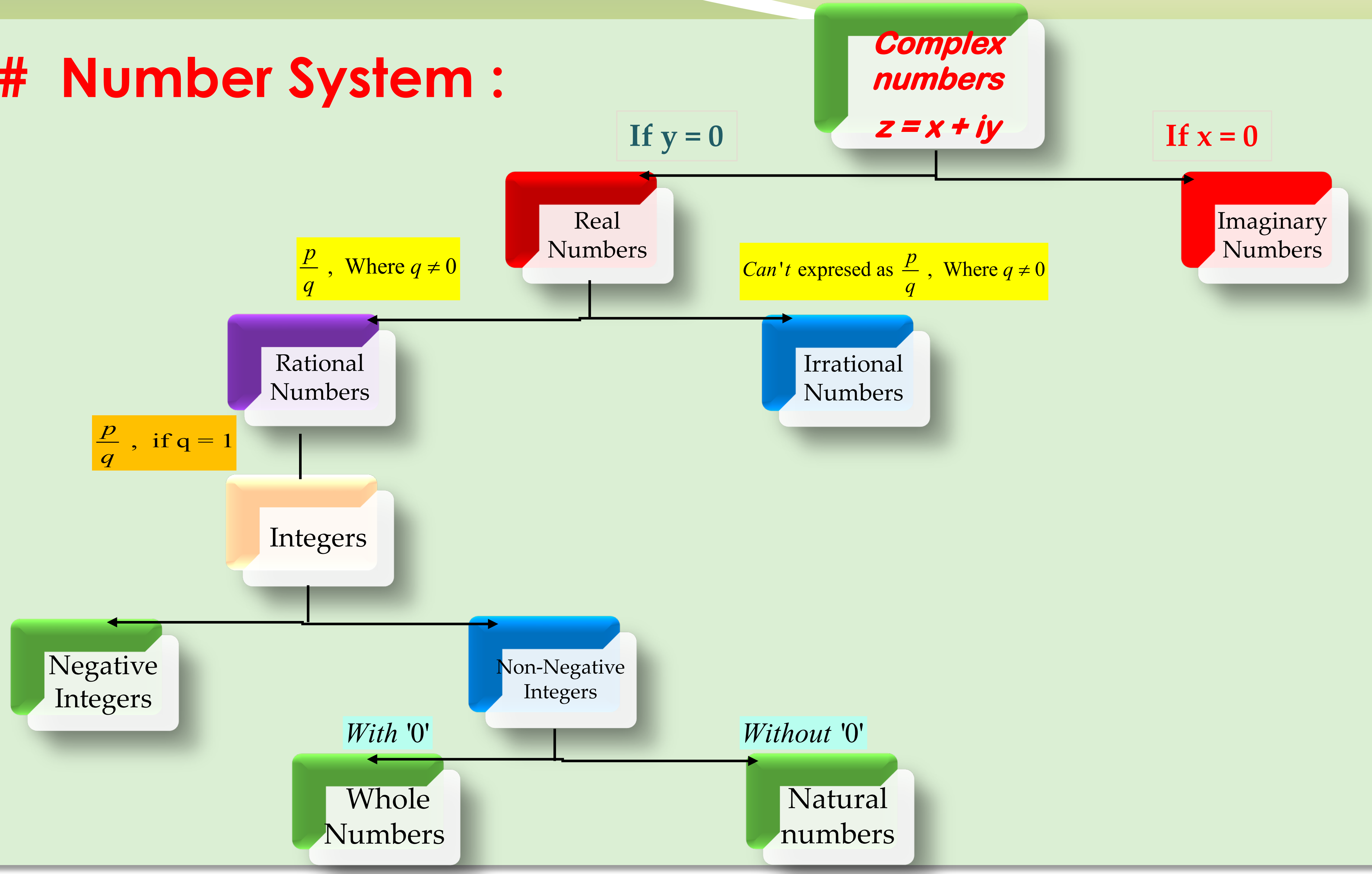


Re(z) = a
Im(z) = b

Why are these numbers important?

- So why do we study complex numbers anyway? Believe it or not, complex numbers have many applications—electrical engineering and quantum mechanics to name a few!
- From a purely mathematical standpoint, one cool thing that complex numbers allow us to do is to solve any polynomial equation.
- For example, the polynomial equation $x^2 - 2x + 5 = 0$ does not have any real solutions nor any imaginary solutions. However, it does have two complex number solutions. These are 1+2i, and 1- 2i.
- As we continue our study of mathematics, we will learn more about these numbers and where they are used.

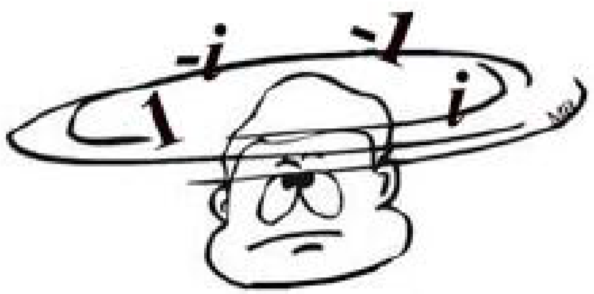
Number System :



Power of *i* -

When the imaginary unit, *i*, is raised to increasingly higher powers, a cyclic (repetitive) pattern emerges.

Remember that $i^2 = -1$.

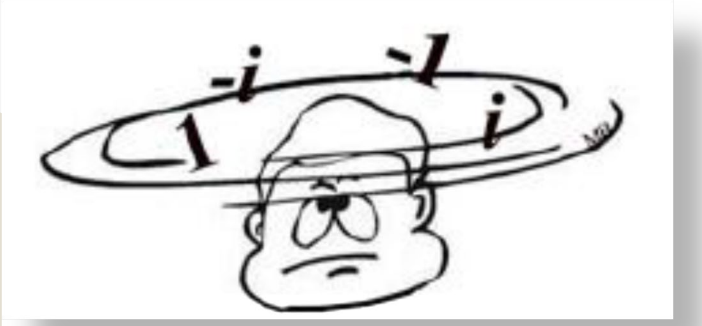


Repeating Pattern of Powers of *i* :		
$i^0 = 1$	$i^4 = i^3 \cdot i = (-i) \cdot i = -i^2 = 1$	$i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$
$i^1 = i$	$i^5 = i^4 \cdot i = 1 \cdot (i) = i$	$i^9 = i^4 \cdot i^4 \cdot i = 1 \cdot 1 \cdot i = i$
$i^2 = -1$	$i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$	$i^{10} = (i^4)^2 \cdot i^2 = 1 \cdot (-1) = -1$
$i^3 = i^2 \cdot i = (-1) \cdot i = -i$	$i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$	$i^{11} = (i^4)^2 \cdot i^3 = 1 \cdot (-i) = -i$

The powers of *i* repeat in a definite pattern: $(i, -1, -i, 1)$

Powers of *i*	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8	...
Simplified Form	i	-1	$-i$	1	i	-1	$-i$	1	...

Power of *i* -



Simplifying powers of i:

You will need to remember (or establish) the powers of ***i*** ***from*** *1 to 4, to obtain one cycle of the pattern. From that list of values, you can easily determine any other positive integer powers of i.*

Method 1: *Divide the exponent by 4:*

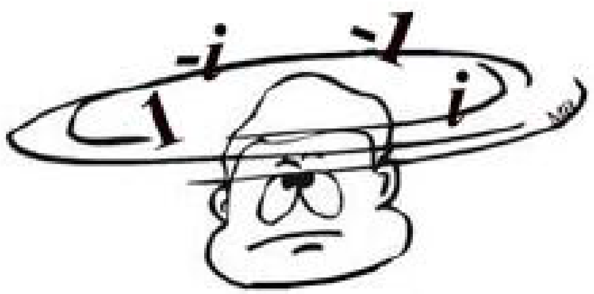
- *if the remainder is 0, the answer is 1* (i^0).
- *if the remainder is 1, the answer is i* (i^1).
- *if the remainder is 2, the answer is -1* (i^2).
- *if the remainder is 3, the answer is -i* (i^3).

- **When raising *i* to any positive integer power, the answer is always *i*, -1, -*i* or 1.**

Method 2: *Divide the exponent by 2 :*

- *if the remainder is 0,* $i^{2n} = \left(i^2\right)^n = \left(-1\right)^n = 1 \text{ or } -1$ based on n is even or odd
- *if the remainder is 1,* $i^{2n+1} = \left(i^{2n}\right)i = \left(-1\right)^n i = i \text{ or } -i$ based on n is even or odd

Example 1 - Simplify i^{87}



Answer : By Method 1

Divide the power by 4 to find the remainder.

$87 \div 4 = 21$ with remainder 3

The answer is $i^{21\times4+3} = \left(i^4\right)^{21} \times i^3 = 1\times(-i) = -i$

By Method 2

Divide the power by 2 to find the remainder.

$87 \div 2 = 43$ with remainder 3

The answer is $i^{43\times2+1} = \left(i^2\right)^{43} \times i = -1\times(i) = -i$

Example 2 - Simplify: $i^{89} + i^{90} + i^{91} + i^{92}$

Answer : Let us simplify each term using Method 2,

$$i^{89} = i^{88} i = \left(i^2\right)^{44} i = (-1)^{44} i = i$$

$$i^{90} = \left(i^2\right)^{45} = (-1)^{45} = -1$$

$$i^{91} = i^{90} i = \left(i^2\right)^{45} i = (-1)^{45} i = -i$$

$$i^{92} = \left(i^2\right)^{46} = (-1)^{46} = 1$$

Adding all the terms we get

$$i^{89} + i^{90} + i^{91} + i^{92} = \cancel{i} - \cancel{1} - \cancel{i} + \cancel{1} = 0$$

Note - In general if we take four consecutive powers of i and add the answer is always 0

i.e. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i - 1 - i + 1 = 0$

Example 3 - Simplify: $\sum_{k=1}^{13}\left(i^k+i^{k+1}\right)$

Answer :

$$\sum_{k=1}^{13}\left(i^k+i^{k+1}\right)=\sum_{k=1}^{13} i^k+\sum_{k=1}^{13} i^{k+1}$$

$$=\left(i^1+i^2+i^3+i^4+i^5+.........+i^{13}\right)+\left(i^2+i^3+i^4+i^5+i^6+..................+i^{14}\right)$$

$$=i^1+i^2=i-1$$

Question for Practice : Simplify - $\sum_{k=1}^{2020}\left(i^k+i^{k+1}\right)$ Ans: i

Equality of two complex numbers -

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if and only if a = c and b = d i.e., $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

For example, if the complex numbers z_1 = x + iy and z_2 = -5 + 7i are equal, then x = -5 & y = 7.

Example 4. If a, b are real numbers and $7a + i(3a - b) = 14 - 6i$, then find the values of a and b.

Solution:

Given, $7a + i(3a - b) = 14 - 6i$

$\Rightarrow 7a + i(3a - b) = 14 + i(-6)$

Now equating real and imaginary parts on both sides, we have

$7a = 14$ and $3a - b = -6$

$\Rightarrow a = 2$ and $3 \cdot 2 - b = -6$

$\Rightarrow a = 2$ and $6 - b = -6$

$\Rightarrow a = 2$ and $-b = -12$

$\Rightarrow a = 2$ and $b = 12$

Therefore, the value of a = 2 and the value of b = 12.

Modulus of complex numbers -

Let $z = x + iy$ where x and y are real and $i = \sqrt{-1}$.

Then the non negative square root of $\left(x^2 + y^2\right)$ is called the modulus or absolute value of z (or $x + iy$).

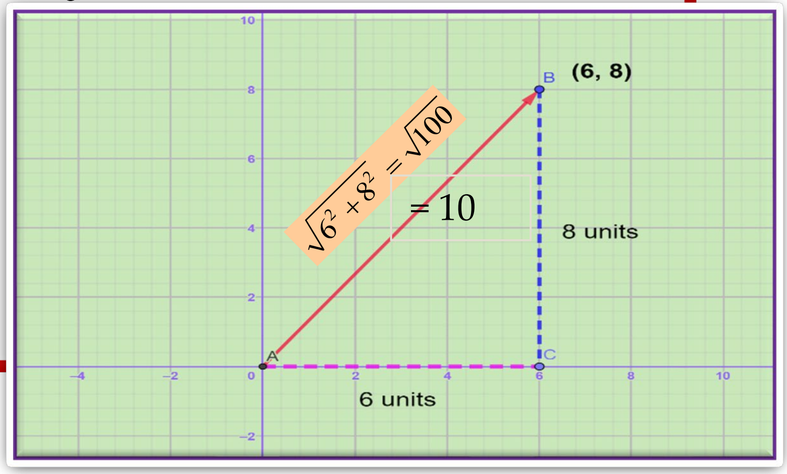
Modulus of a complex number $z = x + iy$, denoted by mod(z) or $|z|$ or $|x + iy|$, is defined as $|z| = +\sqrt{x^2 + y^2} = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}$

Sometimes, $|z|$ is called absolute value of z. Clearly, $|z| \geq 0$ for all $z \in C$.

For example 5 :

(i) If $z = 6 + 8i$ then $|z| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.

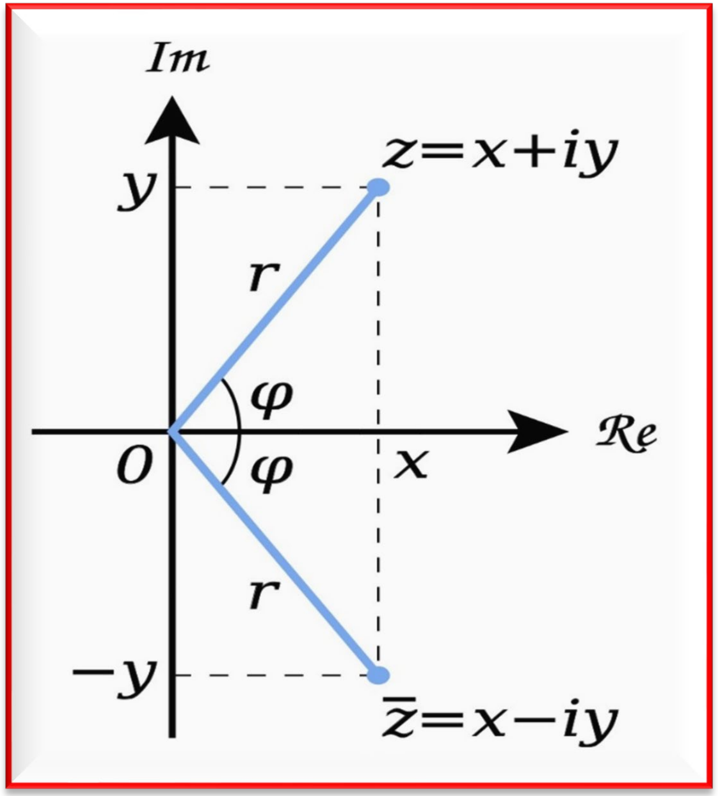
(ii) If $z = -6 + 8i$ then $|z| = \sqrt{(-6)^2 + 8^2} = \sqrt{100} = 10$.



Conjugate of complex numbers -

The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude, but opposite in sign.

Given a complex number $z = x + iy$, the complex conjugate of z, is denoted by $\bar{z}$ and is equal to $\bar{z} = x - iy$ see fig.



Example 6.
To find the complex conjugate of 4+7i we change the sign of the imaginary part. Thus the complex conjugate of 4+7i is **4 – 7i.**

Example 7.
To find the complex conjugate of 1–3i we change the sign of the imaginary part. Thus the complex conjugate of 1 – 3i is **1+3i.**

Example 8.
To find the complex conjugate of –4 – 3i we change the sign of the imaginary part. Thus the complex conjugate of –4 – 3i is **–4 + 3i.**

Properties of Modulus & Conjugate of complex numbers -

1. $|z_1 z_2| = |z_1|.|z_2|$
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, where $z_2 \neq 0$
3. $z.\overline{z} = |z|^2$
4. $\overline{\overline{z}} = z$
5. $|z| = |\overline{z}|$
6. $\overline{(z_1 \pm z_2)} = \overline{z}_1 \pm \overline{z}_2$
7. $\overline{(z_1 z_2)} = \overline{z}_1 . \overline{z}_2$
8. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$, where $z_2 \neq 0$

1. $|z| = 0 \Leftrightarrow z = 0$ i.e., $\mathrm{Re}\,(z) = 0$ and $\mathrm{Im}\,(z) = 0$
2. $|z| = |\overline{z}| = |-z|$
3. $-|z| \leq \mathrm{Re}\,(z) \leq |z|$ and $-|z| \leq \mathrm{Im}\,(z) \leq |z|$
4. $z\,\overline{z} = |z|^2$, $|z^2| = |\overline{z}|^2$
5. $|z_1 z_2| = |z_1||z_2|$, $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$
6. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\,(z_1\,\overline{z}_2)$
7. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}\,(z_1\,\overline{z}_2)$
8. $|z_1 + z_2| \leq |z_1| + |z_2|$
9. $|z_1 - z_2| \geq |z_1| - |z_2|$
10. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

    In particular:

    $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$
11. As stated earlier multiplicative inverse (reciprocal) of a complex number $z = a + ib$ $(\neq 0)$ is

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\overline{z}}{|z|^2}$$

Happy Learning