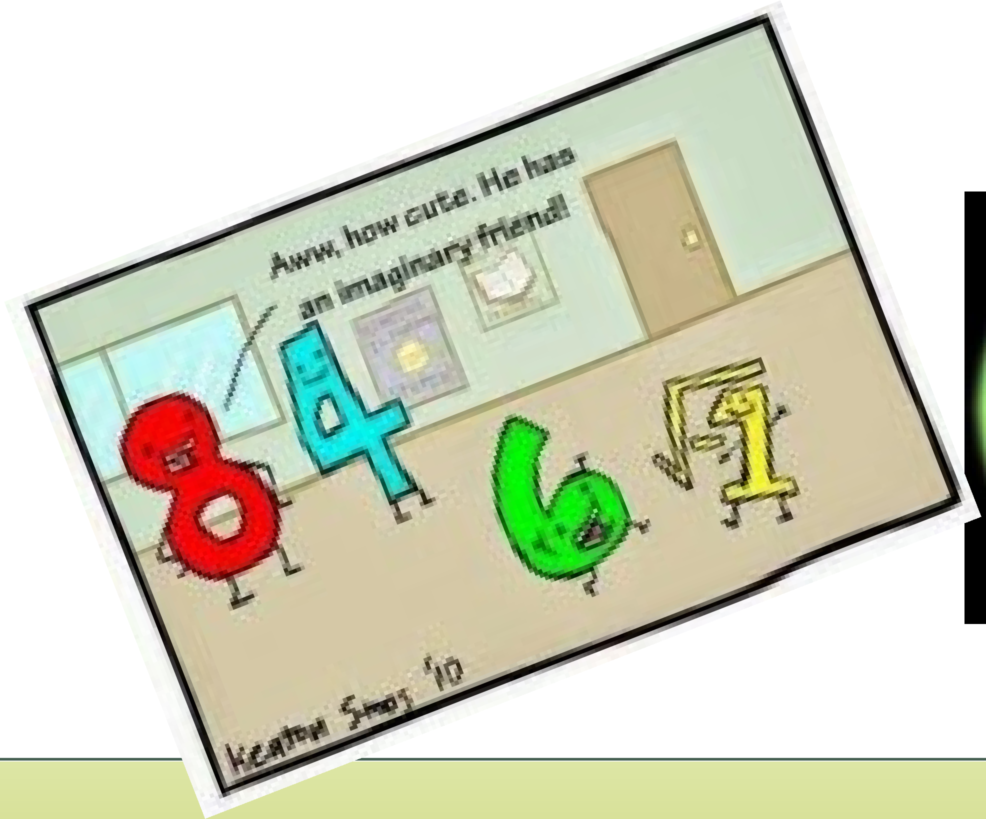
परमाणु ऊर्जा शिक्षण संस्था
(परमाणु ऊर्जा विभाग का स्वायत्त निकाय, भारत सरकार)
ATOMIC ENERGY EDUCATION SOCIETY
(An autonomous body under Department of Atomic Energy, Govt. of India)

5. Complex Numbers & Quadratic Equations



Module II

e-content

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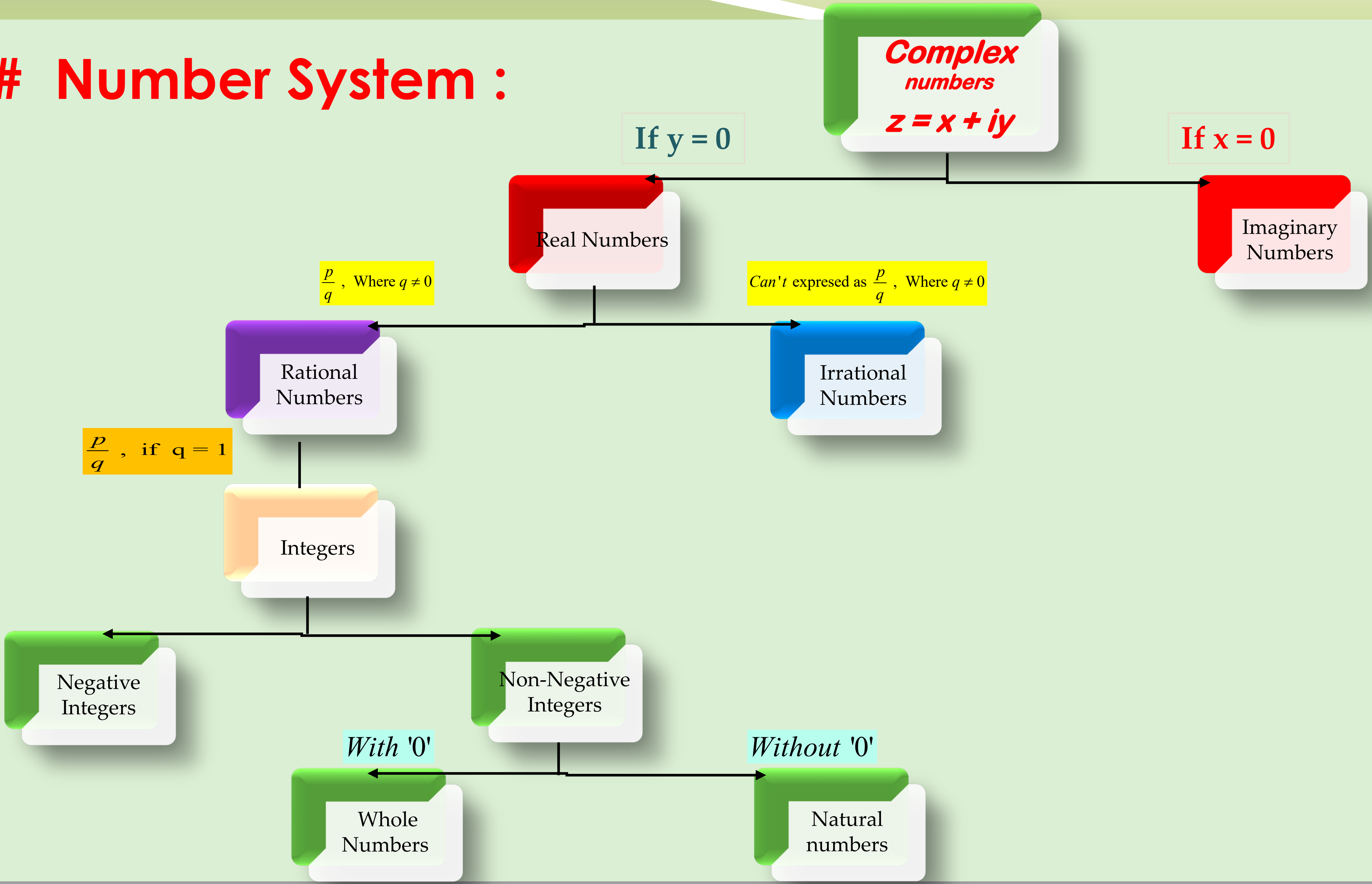
\# Number System :



Addition and Difference of two Complex Numbers :

When adding complex numbers, add the real parts together and add the imaginary parts together. i.e. If $z_1 = a + ib$ and $z_2 = c + id$ then, $z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d)$

Example:

$$if\ \ z_1 = 3 + 7i \ \text{ and } \ z_2 = 8 + 11i$$

$$then, \qquad z_1 + z_2 =$$

imaginary part

(3 + 7i) + (8 + 11i)

real part

$z_1 + z_2 =$ $\boxed{11 + 18i}$

When subtracting complex numbers, be sure to distribute the subtraction sign; then add like parts.

If $z_1 = a + ib$ and $z_2 = c + id$ then, $z_1 - z_2 = (a+ib) - (c+id) = (a-c) + i(b-d)$

Example: $if\ z_1 = 5 + 10i$ and $z_2 = 15 - 2i$

$then,$ find $z_1 - z_2$.

$$\begin{aligned} z_1 - z_2 &= (5 + 10i) - (15 - 2i) \\ &= 5 + 10i - 15 + 2i \\ &= \boxed{-10 + 12i} \end{aligned}$$

#Properties of Addition of complex nos. :

The addition of complex numbers satisfy the following properties:

(i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number*, such that, for every complex number z, $z + 0 = z$.

(v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of z*. We observe that $z + (-z) = 0$ (the additive identity).

Multiplication of two Complex Numbers :

When multiplying complex numbers,
use the distributive property and simplify.

If $z_1 = a + ib$ and $z_2 = c + id$ then, $z_1 z_2 = (a + ib) \times (c + id) = (ac - bd) + i(ad + bc)$

Example: $if\ \ z_1 = 3 - 8i$ and $z_2 = 5 + 7i$

$then,$ find $z_1 z_2$.

$$z_1 z_2 = (3 - 8i)(5 + 7i)$$

$$= 15 + 21i - 40i - 56i^2$$

$$= 15 - 19i + 56$$

$$z_1 z_2 = \boxed{71 - 19i}$$

Remember,
$i^2 = -1$

Properties of Multiplication of Complex Numbers :

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1\ z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,

$$z_1\ z_2 = z_2\ z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,

$$(z_1\ z_2)\ z_3 = z_1\ (z_2\ z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi(a \neq 0,\ b \neq 0)$, we have the complex number

$\dfrac{a}{a^2+b^2} + i\dfrac{-b}{a^2+b^2}$ (denoted by $\dfrac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$$z.\frac{1}{z} = 1 \text{ (the multiplicative identity).}$$

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,

(a) $z_1\ (z_2 + z_3) = z_1\ z_2 + z_1\ z_3$

(b) $(z_1 + z_2)\ z_3 = z_1\ z_3 + z_2\ z_3$

Division of two Complex Numbers :

To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the complex number in the denominator of the fraction. i.e

If $z_1 = a + ib$ and $z_2 = c + id$ then $\frac{z_1}{z_2} = \frac{a+ib}{c+id}$ where $z_2 \neq 0$

Which is further simplified as

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \quad \text{as } \overline{z}_2 = c - id$$

$$= \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Example:

$$if \ z_1 = 7 + 2i \ \text{ and } \ z_2 = 3 - 5i$$

$$then, \ \text{find} \ \frac{z_1}{z_2}.$$

$$\frac{7 + 2i}{3 - 5i} = \frac{7 + 2i}{3 - 5i} \frac{(3 + 5i)}{(3 + 5i)}$$

As The complex conjugate of 3 – 5i is 3 + 5i.

$$= \frac{21 + 35i + 6i + 10i^2}{9 + 15i - 15i - 25i^2}$$

$$= \frac{21 + 41i - 10}{9 + 25} = \frac{11 + 41i}{34}$$

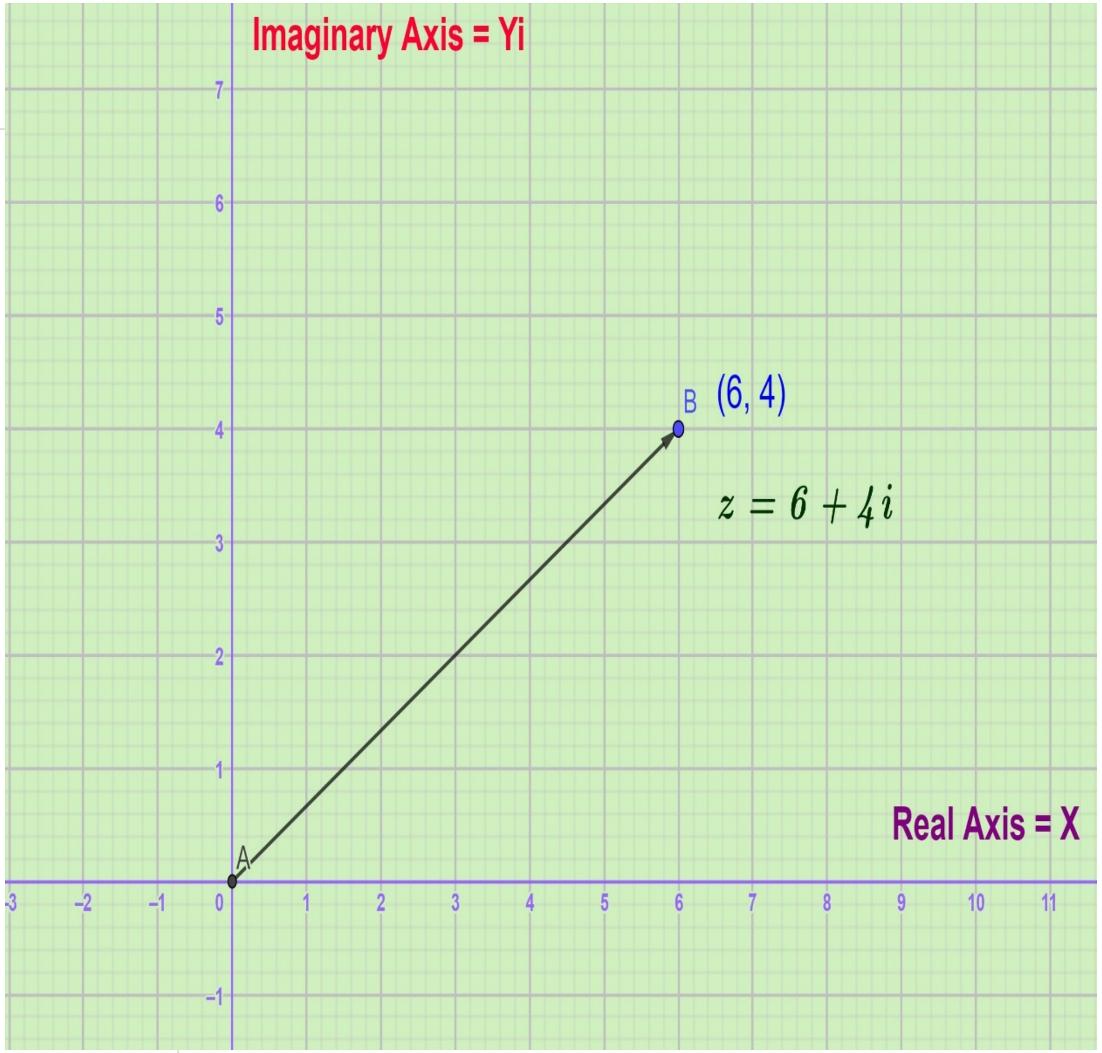
Try These.

1. $(3 + 5i) – (11 – 9i)$ Ans. $-8 + 14i$
2. $(5 – 6i)(2 + 7i)$ $52 + 23i$
3. $\dfrac{2 – 3i}{5 + 8i}$ $\dfrac{–14 – 31i}{89}$
4. $(19 – i) + (4 + 15i)$ $23 + 14i$

Argand Plane and Polar Representation:

Argand Plane

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa. Like from the fig. Coordinates of pt. B(6, 4) then the Complex No. will be z = 6 + 4i.



Some complex numbers such as $2 + 4i$, $-2 + 3i$, $0 + 1i$, $2 + 0i$, $-5 - 2i$ and $1 - 2i$ which correspond to the ordered pairs $(2, 4)$, $(-2, 3)$, $(0, 1)$, $(2, 0)$, $(-5, -2)$, and $(1, -2)$, respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 5.1.

The plane having a complex number assigned to each of its point is called the ***Complex plane* or the *Argand plane***

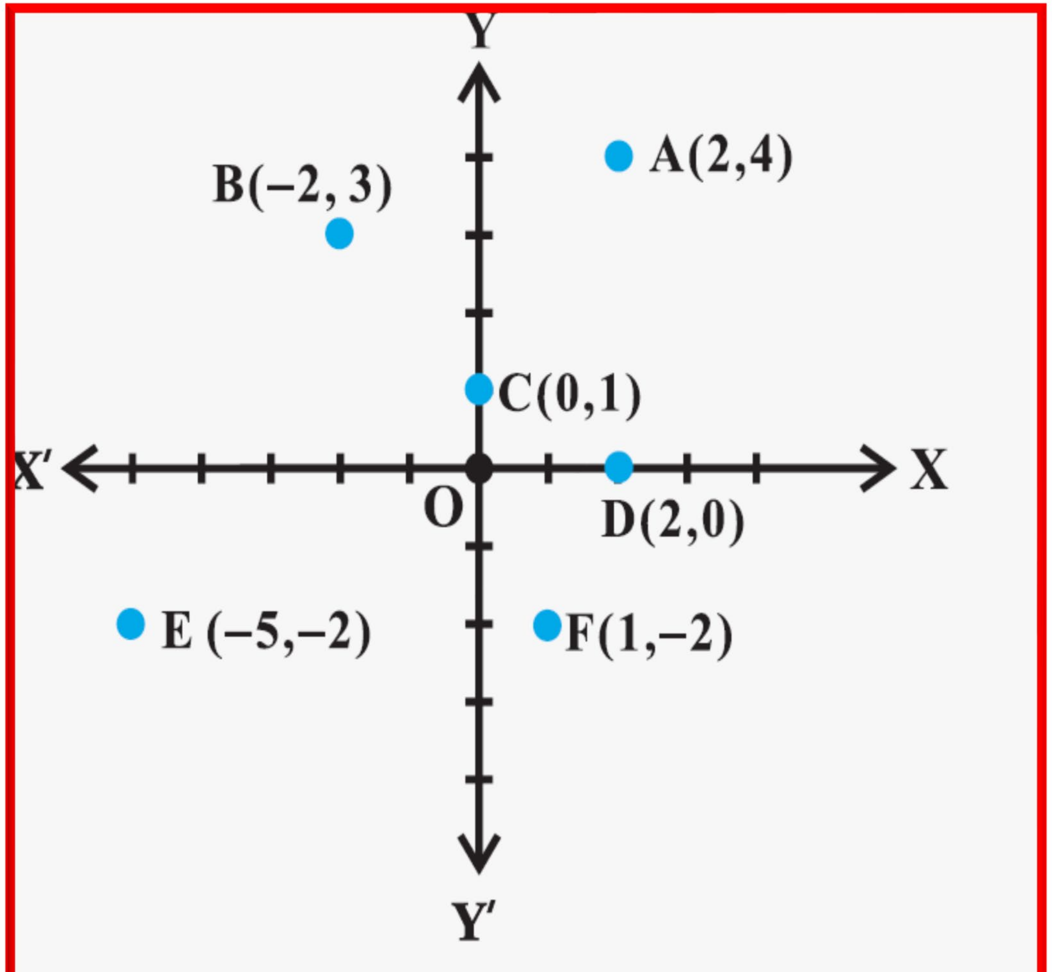


Fig 5.1

Polar representation of a complex number

Let the point P represent the nonzero complex number $z = x + iy$. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis (Fig 5.4).
We may note that the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the *polar coordinates of the point* P. We consider the origin as the pole and the positive direction of the x axis as the initial line.
We have, $x = r \cos \theta$, $y = r \sin \theta$

therefore, $z = r (\cos \theta + i \sin \theta)$.
is said to be the *polar form of the complex number*.

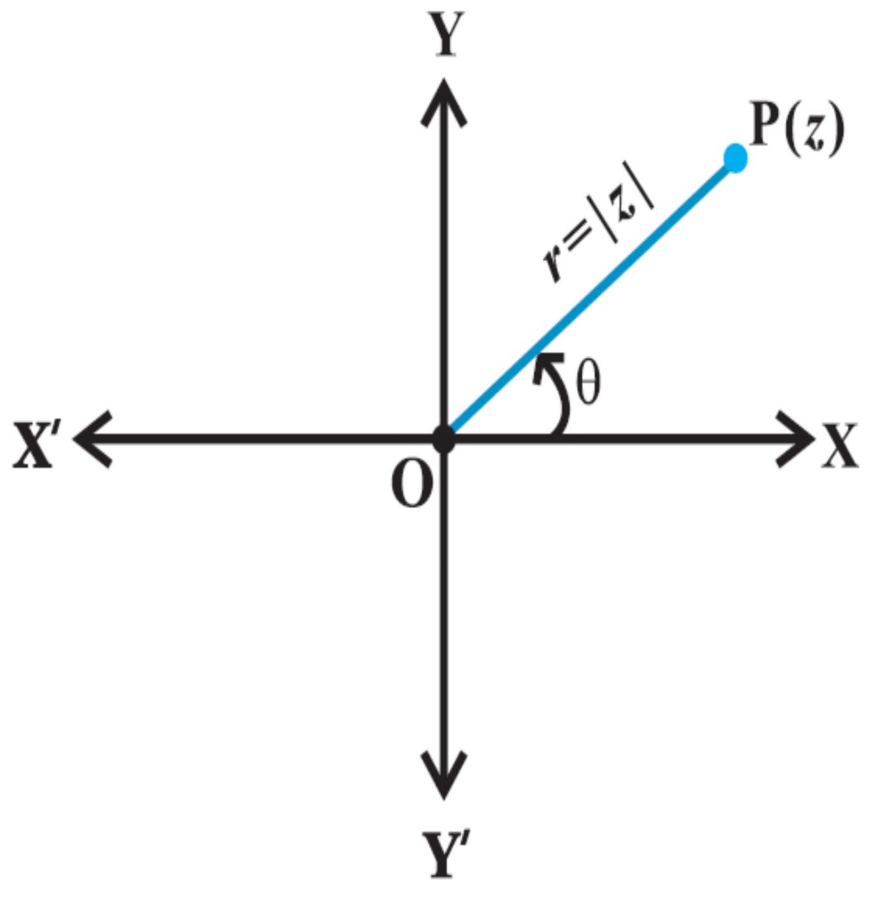


Fig 5.4

Here $r = \sqrt{x^2 + y^2} = |z|$ is the **modulus of** z

and θ is called the argument (or amplitude) of z which is denoted by arg z.

For any complex number $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be such an interval. We shall take the value of θ such that

$-\pi < \theta \leq \pi$, called principal argument of z and is denoted by arg z,

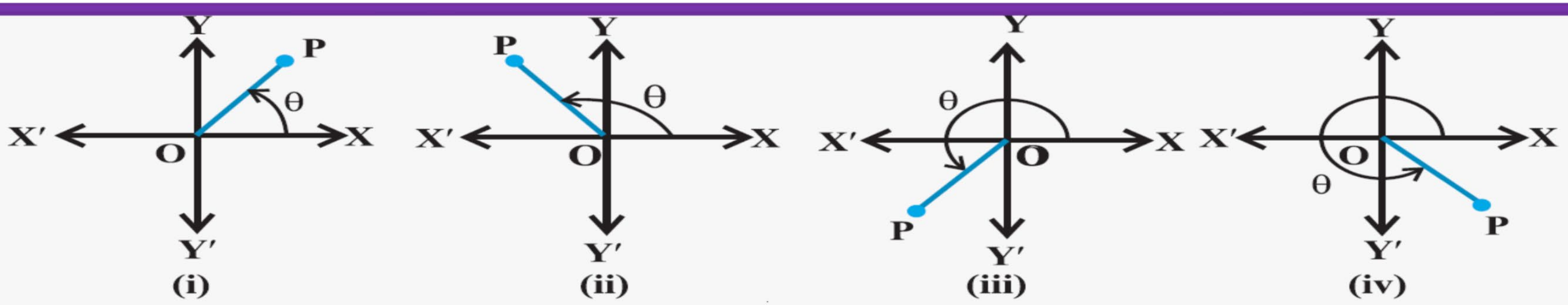


Fig 5.5 $(0 \leq \theta < 2\pi)$

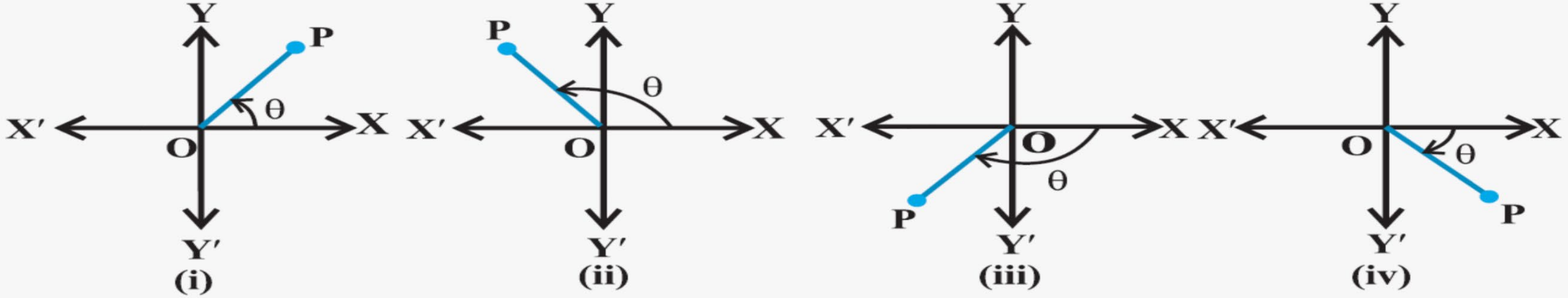


Fig 5.6 $(-\pi < \theta \leq \pi)$

Algorithm to convert Cartesian form into Polar form of a complex number.

If $z = x + iy$ *is cartesian form of complex number then, use the following steps to convent it into polar form*

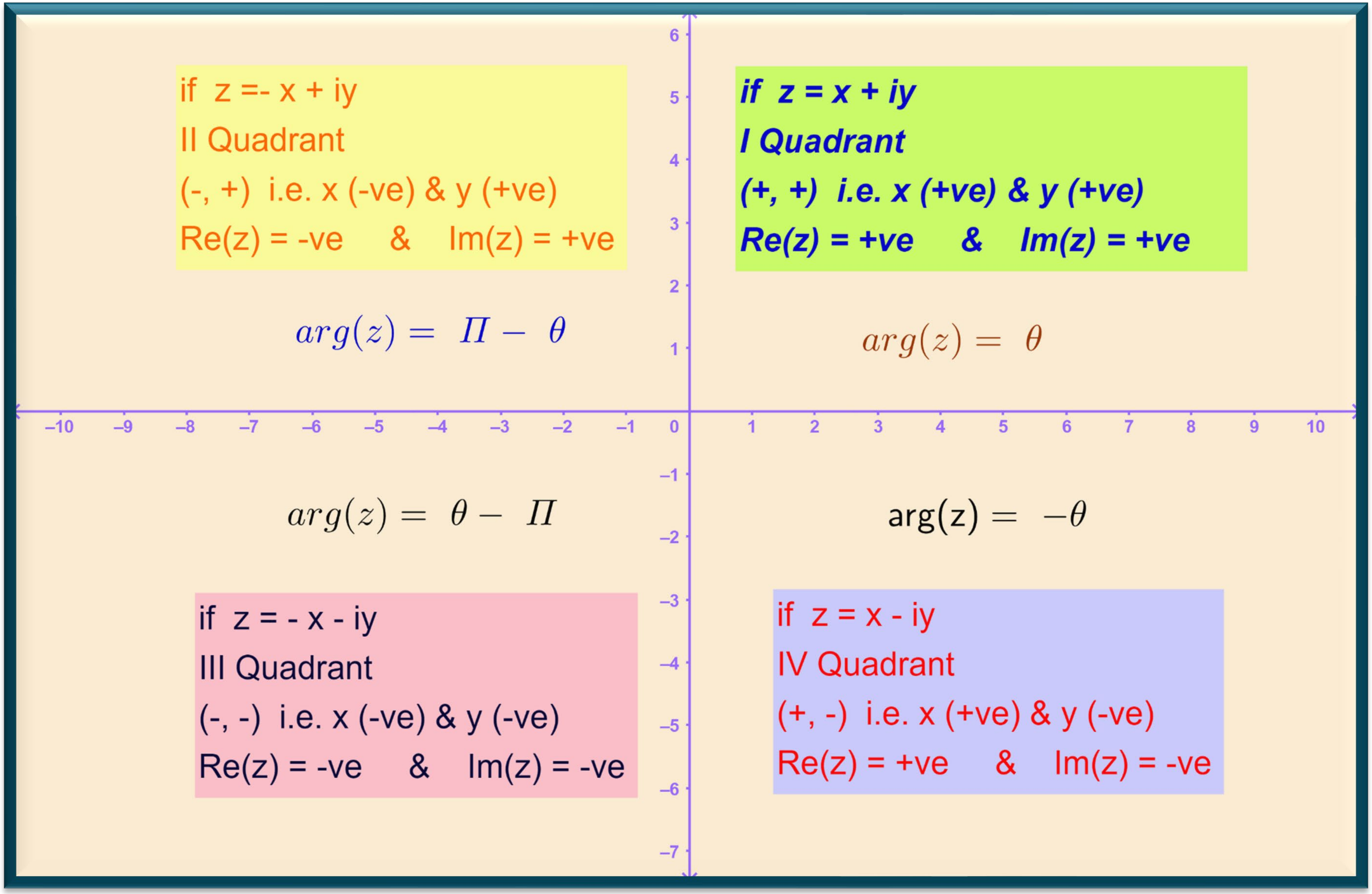
1. Find the modulus of z as $r = \sqrt{x^2 + y^2}$

2. Find the argument of z as, $\tan\theta = \dfrac{y}{x}$ *and based on quadrant find the arg(z), use the graph given in the next slide.*

3. Write complex number z as, $\boxed{z = r(\cos\theta + i\sin\theta)}$ *which is the required polar form of the given complex number.*

4. Remember: $\boxed{\cos(-\theta) = \cos(\theta) \quad \& \quad \sin(-\theta) = -\sin(\theta)}$

To Find Argument Remember this Graph -



Example: Convert the complex number $\dfrac{-16}{1+i\sqrt{3}}$ into polar form.

Solution- The given complex number $\dfrac{-16}{1+i\sqrt{3}} = \dfrac{-16}{1+i\sqrt{3}} \times \dfrac{1-i\sqrt{3}}{1-i\sqrt{3}}$

$$= \frac{-16\left(1-i\sqrt{3}\right)}{1-\left(i\sqrt{3}\right)^2} = \frac{-16\left(1-i\sqrt{3}\right)}{1+3} = -4\left(1-i\sqrt{3}\right) = -4+i4\sqrt{3} \text{ (Fig 5.8).}$$

$$Now,\quad |z| = r = \sqrt{(-4)^2+(4\sqrt{3})^2} = 8$$

$$\tan\theta = \frac{y}{x} = \frac{4\sqrt{3}}{-4} = -\sqrt{3}$$

as x is -ve and y is +ve θ lie in II quadrant

$$\therefore \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Thus reqired polar form is $z = 8\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

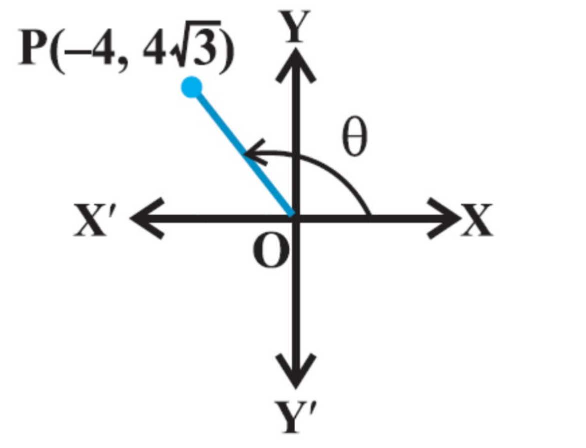


Fig 5.8

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