Class XI- MATHEMATICS

Chapter-3 : TRIGONOMETRIC FUNCTIONS

Hand out of Module 3/3

<u>Learning Outcome</u>:

In this module we are going to learn about

- **Trigonometric Functions of Sum and Difference of Two Angles.**
- **Representation of T-ratios of multiples of an angle in terms of T-ratios of an angle.**
- **Representation of product of T-ratios as sum or difference of T-ratios.**
- **Representation of sum or difference of T-ratios as product of T-ratios.**

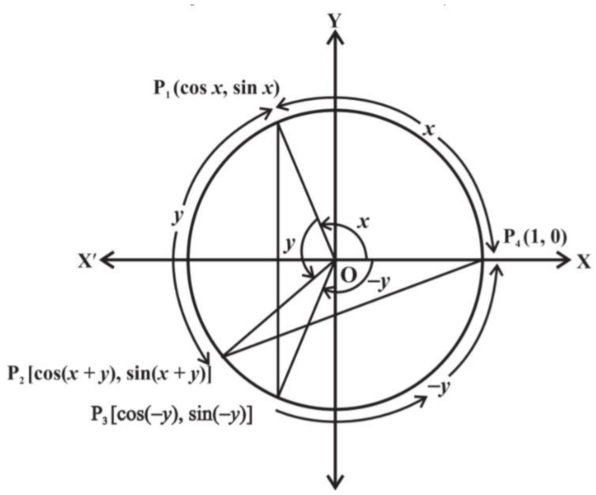
<u>Trigonometric Functions of Sum and Difference of Two Angles.</u>

1). cos (x + y) = cos x cos y – sin x sin y

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2 . Then (x + y) is the angle P_4OP_2 . Also let (– y) be the angle P_4OP_3 . Therefore, P_1, P_2, P_3 and P_4 will have the coordinates P_1 (cos x, sin x), P_2 [cos (x + y), sin (x + y)], P_3 [cos (– y), sin (– y)] and P_4 (1, 0).

Triangles P_1OP_3 and P_2OP_4 are congruent .

Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula we get ,



$P_1P_3^2$ = [cos x – cos (– y)]^2 + [sin x – sin(–y]^2

= (cos x – cos y) ^{2} + (sin x + sin y) ^{2}

= cos^2 x + cos^2 y – 2 cos x cos y + sin^2 x + sin^2 y + 2sin x sin y

= 2 – 2 (cos x cos y – sin x sin y) ……(1)

Also, $P_2P_4^2$ = [1 – cos (x + y)] ^{2} + [0 – sin (x + y)]^2

= 1 – 2cos (x + y) + cos^2 (x + y) + sin^2 (x + y)

= 2 – 2 cos (x + y)……………….(2)

From (1) and (2) we get,

2 – 2 (cos x cos y – sin x sin y) = 2 – 2 cos (x + y)

– 2 (cos x cos y – sin x sin y) = – 2 cos (x + y)

Or, **cos (x + y) = cos x cos y – sin x sin y**

Similarly we can prove,

2) **cos (x – y) = cos x cos y + sin x sin y**

3) **sin (x + y) = sin x cos y + cos x sin y**

4) **sin (x - y) = sin x cos y - cos x sin y**

5). By using the above formulae, we can prove that

i	$\cos\left(\frac{\pi}{2} - x\right) = \sin x$	**ix**	$\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$
ii	$\sin\left(\frac{\pi}{2} - x\right) = \cos x$	**x**	$\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
iii	$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$	**xi**	$\cos\left(\frac{3\pi}{2} + x\right) = \sin x$
iv	$\sin\left(\frac{\pi}{2} + x\right) = \cos x$	**xii**	$\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$
v	$\text{Cos}(\pi - x) = -\cos x$	**xiii**	$\text{Cos}(2\pi - x) = \cos x$
vi	$\text{Sin}(\pi - x) = \sin x$	**xiv**	$\text{Sin}(2\pi - x) = -\sin x$
vii	$\text{Cos}(\pi + x) = -\cos x$	**xv**	$\text{Cos}(2\pi + x) = \cos x$
viii	$\text{Sin}(\pi + x) = -\sin x$	**xvi**	$\text{Sin}(2\pi + x) = \sin x$

6). If none of the angles x, y and (x + y) is an odd multiple of $\frac{\pi}{2}$, then

i) **tan (x + y)** = $\frac{\tan x + \tan y}{1 - \tan x \tan y}$

ii) **tan (x - y)** = $\frac{\tan x - \tan y}{1 + \tan x \tan y}$

Proof : i) tan (x + y) = $\frac{\sin(x+y)}{\cos(x+y)}$

$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$ (on dividing numerator and denominator by cos x cos y)

Replacing y by -y we can prove that , **tan (x - y)** = $\frac{\tan x - \tan y}{1 + \tan x \tan y}$

7). **If none of the angles *x*, *y* and (*x* + *y*) is a multiple of π, then**

i) cot (x + y) = $\frac{\cot x \cot y - 1}{\cot y + \cot x}$

ii) cot (x - y) = $\frac{\cot x \cot y + 1}{\cot y - \cot x}$

Proof: $\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}$

$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

$= \frac{\cot x \cot y - 1}{\cot y + \cot x}$ (on dividing numerator and denominator by sin x sin y)

ii) Replacing y by -y we can prove that , $\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

Representation of T-ratios of multiples of an angle in terms of T-ratios of an angle.

i). $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

Proof. $\cos 2x = \cos(x+x) = \cos x . \cos x - \sin x . \sin x$

$= \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$

$= 2(1 - \sin^2 x) - 1$

$= 1 - 2\sin^2 x$

Also, $\cos 2x = \cos^2 x - \sin^2 x$

(on dividing numerator and denominator by $\cos^2 x$)

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

ii) $\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

Proof: $\sin 2x = \sin(x+x) = \sin x \cos x + \cos x \sin x$

$= 2\sin x \cos x$

$= \frac{2 \sin x \cos x}{1}$

$= \frac{2 \sin x \cos x}{\cos^2 x + \sin^2 x}$

$= \frac{2\tan x}{1+\tan^2 x}$ (on dividing numerator and denominator by $\cos^2 x$)

iii) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

Proof: $\tan 2x = \frac{\sin 2x}{\cos 2x}$

$$= \frac{\frac{2\tan x}{1+\tan^2 x}}{\frac{1-\tan^2 x}{1+\tan^2 x}}$$

$$= \frac{2\tan x}{1-\tan^2 x}$$

Similarly we can prove,

iv). **$\sin 3x = 3\sin x - 4\sin^3 x$**

v) **$\cos 3x = 4\cos^3 x - 3\cos x$**

vi) **$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$**

Representation of product of T-ratios as sum or difference of T-ratios.

1) **$2\cos x \cos y = \cos(x + y) + \cos(x - y)$**
2) **$-2\sin x \sin y = \cos(x + y) - \cos(x - y)$**
3) **$2\sin x \cos y = \sin(x + y) + \sin(x - y)$**
4) **$2\cos x \sin y = \sin(x + y) - \sin(x - y)$.**

Proof:

We know that

$\cos(x + y) = \cos x \cos y - \sin x \sin y$ ... (1)

and $\cos(x - y) = \cos x \cos y + \sin x \sin y$ ... (2)

Adding and subtracting (1) and (2), we get

$\cos(x + y) + \cos(x - y) = 2\cos x \cos y$

and $\cos(x + y) - \cos(x - y) = -2\sin x \sin y$

Hence, **$2\cos x \cos y = \cos(x + y) + \cos(x - y)$**

$-2\sin x \sin y = \cos(x + y) - \cos(x - y)$

Similarly by using sine formula ,we can prove

$2\sin x \cos y = \sin(x + y) + \sin(x - y)$

$2\cos x \sin y = \sin(x + y) - \sin(x - y)$.

Representation of sum or difference of T-ratios as product of T-ratios

1) $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

2) $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

3) $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

4) $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

We have proved that

$\cos(x + y) + \cos(x - y) = 2\cos x \cos y$ ……………(i)
$\cos(x + y) - \cos(x - y) = -2\sin x \sin y$……………..(ii)

Let $x + y = \theta$ and $x - y = \emptyset$, then $x = \frac{\theta + \emptyset}{2}$ and $y = \frac{\theta - \emptyset}{2}$ ……………..(5)

Substituting the value of x and y in equations (i) and (ii), we get

$\cos\theta + \cos\emptyset = 2\cos\left(\frac{\theta+\emptyset}{2}\right)\ \cos\left(\frac{\theta-\emptyset}{2}\right)$)... (6)

and $\cos\theta - \cos\emptyset = -2\sin\left(\frac{\theta+\emptyset}{2}\right)\ \sin\left(\frac{\theta-\emptyset}{2}\right)$ ....(7)

Since θ and $\emptyset$ can take any real values, we can replace θ by x and $\emptyset$ by y.

Hence, $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Similarly we can prove,

$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\ \sin\left(\frac{x-y}{2}\right)$

Example 1:

Find the value of sin 75°.

Solution: We have, $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Example 2:

Show that, tan 3x tan 2x tan x = tan 3x – tan 2x – tan x

Proof: tan 3x = tan (2x + x)

$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

or, $\tan 3x(1 - \tan 2x \tan x) = \tan 2x + \tan x$

or, tan 3x – tan 3x tan 2x tan x = tan 2x + tan x

or, tan 3x – tan 2x – tan x = tan 3x tan 2x tan x

or, tan 3x tan 2x tan x = tan 3x – tan 2x – tan x.

Example 3:

Prove that $\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x}$ = cot x.

Proof : $\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \frac{2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)}{2\cos\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}$

$= \frac{2\cos 6x \,.\, \cos x}{2\cos 6x \,.\, \sin x} = \cot x$

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