Atomic Energy Education Society, Mumbai

Class XI Subject-Physics

Chapter–4: Motion in a Plane

Module 2/2

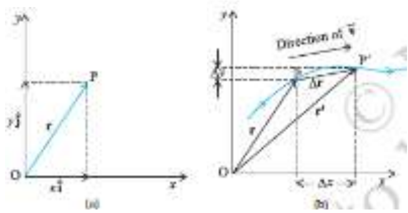
CONTENTS

MOTION IN A PLANE:

Position Vector and Displacement:

The position vector r of a particle P located in a plane with reference to the origin of an *x-y* reference frame (Fig. given below) is given by

$$\mathbf{r} = x\,\mathbf{\hat{i}} + y\,\mathbf{\hat{j}}$$



where *x* and *y* are components of r along *x*-, and *y*- axes or simply they are the coordinates of the object. Suppose a particle moves along the curve shown by the thick line and is at P at time *t* and P′ at time *t*′ [Fig. (b)]. Then, the displacement is :

$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$ and is directed from P to P′.

We can write Eq. (4.25) in a component form:

$$\Delta\mathbf{r} = (x'\,\mathbf{\hat{i}} + y'\,\mathbf{\hat{j}}) - (x\,\mathbf{\hat{i}} + y\,\mathbf{\hat{j}})$$
$$= \mathbf{\hat{i}}\Delta x + \mathbf{\hat{j}}\Delta y$$

where $\Delta x = x' - x,\ \Delta y = y' - y$

Velocity:

The average velocity $\bar{v}$ of an object is the ratio of the displacement and the corresponding time interval :

$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{x}}{\Delta \mathbf{t}} = \frac{\Delta x\boldsymbol{i} + \Delta y\boldsymbol{j}}{\Delta \boldsymbol{t}} = \frac{\Delta x}{\Delta t} i + \frac{\Delta y}{\Delta t} j$$

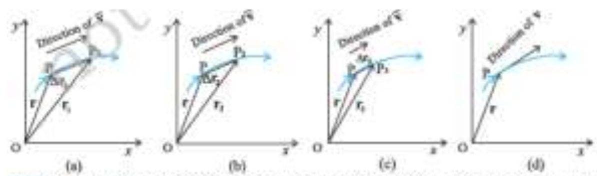
$$\bar{\mathbf{v}} = \vec{\mathbf{v}}_{\mathbf{x}}\mathbf{i} + \vec{\mathbf{v}}_{\mathbf{y}}$$

Since $\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t}$, The direction of average velocity is same as $\Delta \mathbf{r}$

The **velocity** (instantaneous velocity) is given by the limiting value of the average velocity as the time interval approaches zero :

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

The meaning of the limiting process can be easily understood with the help of Fig (a) to (d).



4.13 As the time interval Δt approaches zero, the average velocity approaches the velocity v. The direction of $\bar{\mathbf{v}}$ is parallel to the line tangent to the path.

- In these figures, the thick line represents the path of an object, which is at P at time *t*. P_1, P_2 and P_3 represent the positions of the object after times Δt_1, Δt_2, Δt_3, . Δr_1, Δr_2, , Δr_3, are the displacements of the object in times Δt_1, Δt_2, Δt_3, respectively.
- The direction of the average velocity v is shown in figures (a), (b) and (c) for three decreasing values of Δt_1, Δt_2, Δt_3 ($\Delta t_1 > \Delta t_2 > \Delta t_3$).
- As $\Delta t \to 0$, $\Delta r \to 0$
- And is along the tangent to the path [Fig. (d)].
- **Therefore, the direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion.**

We can express **v i**n a component form :

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

$$= \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}}\right)$$

$$= \hat{\mathbf{i}} \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t}$$

Or, $$\mathbf{v} = \hat{\mathbf{i}}\frac{\mathrm{d}x}{\mathrm{d}t} + \hat{\mathbf{j}}\frac{\mathrm{d}y}{\mathrm{d}t} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}.$$

where $$v_x = \frac{\mathrm{d}x}{\mathrm{d}t}, v_y = \frac{\mathrm{d}y}{\mathrm{d}t}$$

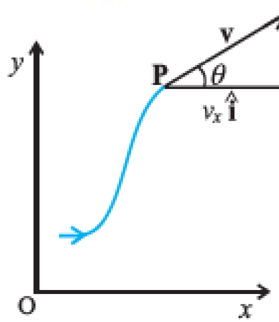
- So, if the expressions for the coordinates *x* and *y* are known as functions of time,
- we can use these equations to find v_x and v_y.

The magnitude of v is then

$$v = \sqrt{v_x^2 + v_y^2}$$

and the direction of **v** is given by the angle θ

$$\tan\theta = \frac{v_y}{v_x}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$



Acceleration:

- The average acceleration a of an object for a time interval Δt moving in *x-y* plane is the change in velocity divided by the time interval :

$$\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta\left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}\right)}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$$

Or, $\bar{\mathbf{a}} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}$.

- The acceleration (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero :

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$$

Since $\Delta \boldsymbol{v} = \Delta v_x \hat{\mathbf{i}} + \Delta v_y \hat{\mathbf{j}}$, we have

$$\mathbf{a} = \hat{\mathbf{i}} \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}$$

Or, $\mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}$

(4.32b)

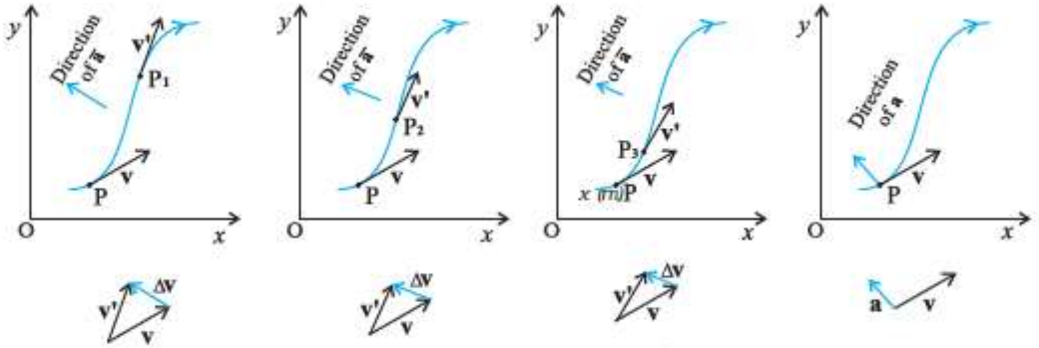
where, $a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$ (4.32c

- As in the case of velocity, we can understand graphically the limiting process used in defining acceleration on a graph showing the path of the object's motion.

- This is shown in the given Figs. (a) to (d).

- P represents the position of the object at time t and P_1 ,P_2 and P_3 position after time Δt_1, Δt_2, and Δt_3 respectively. ($\Delta t_1 > \Delta t_2 > \Delta t_3$).

- The velocity vectors at points P, P_1, P_2, P_3 are also shown in Figs.(a), (b) and (c).



(a) (b) (c) (d)

- In each case of Δt, Δv is obtained using the triangle law of vector addition.
- By definition, the direction of average acceleration is the same as that of Dv.
- We see that as Δt decreases, the direction of Δv changes and consequently, the direction of the acceleration changes.
- Finally, in the limit $\Delta t \rightarrow 0$ [Fig.(d)],
- the average acceleration becomes the instantaneous acceleration and has the direction as shown.
- **Note that in one dimension, the velocity and the acceleration of an object are always along the same straight line (either in the same direction or in the opposite direction).**
- **However, for motion in two or three dimensions, velocity and acceleration vectors may have any angle between 0° and 180° between them.**

MOTION IN A PLANE WITH CONSTANT ACCELERATION

- Suppose that an object is moving in *x-y* plane and its acceleration a is constant.
- Over aninterval of time, the average acceleration will equal this constant value.
- Now, let the velocity of the object be v_0 at time t = 0 and v at time t.
- Then, by definition

$$\mathbf{a} = \frac{\mathbf{v}-\mathbf{v_0}}{t-0} = \frac{\mathbf{v}-\mathbf{v_0}}{t}$$

Or, $\mathbf{v} = \mathbf{v_0} + \mathbf{a}t$

In terms of components :

$$v_x = v_{ox} + a_x t$$

$$v_y = v_{oy} + a_y t$$

- Let us now find how the position r changes with time. We follow the method used in the one dimensional case.
- Let r_o and r be the position vectors of the particle at time 0 and t and
- let the velocities at these instants be v_0 and v.
- Then, over this time interval t, the average velocity is (v_0 + v)/2.
- The displacement is the average velocity multiplied by the time interval :

$$\mathbf{r} - \mathbf{r_0} = \left(\frac{\mathbf{v}+\mathbf{v_0}}{2}\right)t = \left(\frac{(\mathbf{v_0}+\mathbf{a}t)+\mathbf{v_0}}{2}\right)t$$

$$= \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2$$

Or, $\mathbf{r} = \mathbf{r_0} + \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2$

Above Equation (can be written in component form as

$$x = x_0 + v_{ox}t + \frac{1}{2}a_x t^2$$

$$y = y_0 + v_{oy}t + \frac{1}{2}a_y t^2$$

- The motions in *x*- and *y*-directions can be treated independently of each other.
- That is, motion in a plane (two-dimensions) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions.
- . A similar result holds for three dimensions.

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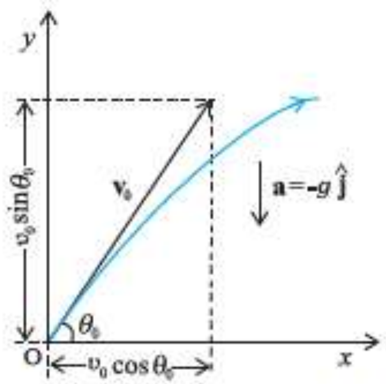
PROJECTILE MOTION:

- . An object that is in flight after being thrown or projected is called a projectile.
- Such a projectile might be a football, a cricket ball, a baseball or any other object.
- The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions.
- One component is along a horizontal direction **without any acceleration** and
- the other along the vertical direction with constant acceleration due to the force of gravity.
- In our discussion, we shall assume that the air resistance has negligible effect on the motion of the projectile.
- Suppose that the projectile is launched with velocity v_0 that makes an angle θ_0 with the *x*-axis as shown in Fig.

.

- After the object has been projected, the acceleration acting on it is that due to gravity which is directed vertically downward: a = $-g$ j



Motion of an object projected with velocity v_0 at angle θ_0

$$\mathbf{a} = -g\,\mathbf{j}$$

Or, $a_x = 0,\ a_y = -g$

The components of initial velocity $\mathbf{v}_o$ are :

$$v_{ox} = v_o \cos\theta_o$$
$$v_{oy} = v_o \sin\theta_o$$

- If we take the initial position to be the origin of the reference frame as shown in Fig.
- , we have :

$$x_o = 0,\ y_o = 0$$

Then, Eq.(4.34b) becomes :

$$x = v_{ox}t = (v_o \cos\theta_o)\,t$$

and $\quad y = (v_o \sin\theta_o)\,t - (\tfrac{1}{2})g\,t^2$

- The components of velocity at time *t* can be obtained:

$$v_x = v_{ox} = v_o \cos\theta_o$$
$$v_y = v_o \sin\theta_o - g\,t$$

- Above equations gives the *x*-, and *y*-coordinates of the position of a projectile at time *t* in terms of two parameters — initial speed v_0 and projection angle θ_0.
- Notice that the choice of mutually perpendicular *x*-, and *y*-directions for the analysis of the projectile motion has resulted in a simplification.
- One of the components of velocity, i.e. *x*-component remains constant throughout the motion
- and only the *y*- component changes, like an object in free fall in vertical direction.
- This is shown graphically at few instants in Fig. .
- Note that at the point of maximum height, $v_y = 0$ and therefore,

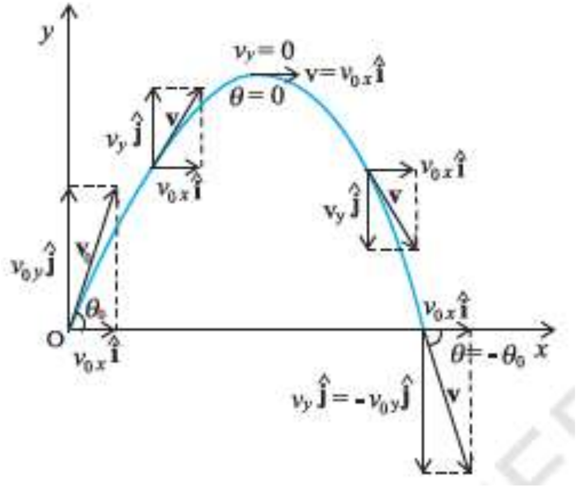
$$\theta = \tan^{-1}\frac{v_y}{v_x} = 0$$

Equation of path of a projectile:

- By eliminating the time between the expressions for x and y as given in Eq. .
- We obtain:

$$y = (\tan\theta_o)x - \frac{g}{2\,(v_o\cos\theta_o)^2}x^2$$

- Now, since g, θ_0 and v_0 are constants,
- Above Eq. is of the form $y = a\,x + b\,x^2$, in which a and b are constants.
- This is the equation of a parabola, i.e. the path of the projectile is a **parabola**



Time Of Maximum Height:

- Let this time be denoted by t_m. Since at this point, $v_y = 0$,
- we have

$$v_y = v_o \sin\theta_o - g\, t_m = 0$$

Or. $$t_m = v_o \sin\theta_o / g$$

:

- The total time T_f during which the projectile is in flight can be obtained by putting $y = 0$ in
- We get :

$$T_f = 2\,(v_o \sin\theta_o)/g$$

- T_f is known as the $T_f = 2\,(v_o \sin\theta_o)/g$ of the projectile.
- We note that $T_f = 2\,t_m$, which is expected because of the symmetry of the parabolic path.

Maximum Height Of A Projectile:

- The maximum height h_m reached by the projectile can be calculated by substituting

$t = t_m$ in Eq. :

$$y = h_m = \left(v_0 \sin\theta_0\right)\left(\frac{v_0 \sin\theta_0}{g}\right) - \frac{g}{2}\left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

Or. $$h_m = \frac{\left(v_0 \sin\theta_0\right)^2}{2g}$$

Horizontal Range Of A Projectile:

- The horizontal distance travelled by a projectile from its initial position ($x = y = 0$) to the position

where it passes $y = 0$ during its fall is called the **horizontal range, *R***.

- It is the distance travelled during the time of flight T_f .
- Therefore, the range R is

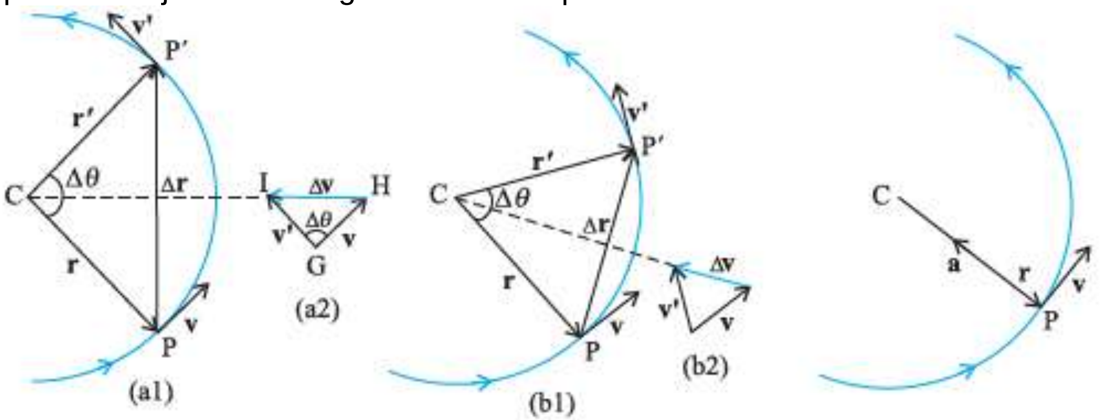
$$R = (v_o \cos \theta_o)(T_f)$$
$$= (v_o \cos \theta_o)\ (2\, v_o \sin \theta_o)/g$$
$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

- Equation shows that for a given projection velocity v_o ,
- R is maximum when $\sin 2\theta_0$ is maximum, i.e., when $\theta_0 = 45^0$.
- The maximum horizontal range is, therefore,

$$R_m = \frac{v_0^2}{g}$$

UNIFORM CIRCULAR MOTION:

- When an object follows a circular path at a constant speed, the motion of the object is called**uniform circular motion**.
- The word “uniform” refers to the speed, which is uniform (constant) throughout the motion.
- Suppose an object is moving with uniform speed *v* in a circle of radius *R* as shown in Fig.



(a) (b) (c)

- . Since the velocity of the object is changing continuously in direction, the object undergoes acceleration.
- Let us find the magnitude and the direction of this acceleration.
- Let r and $\mathbf{r}'$ be the position vectors and v and $\mathbf{v}'$ the velocities of the object when it is at point *P* and P' as shown in Fig.(a).
- By definition, velocity at a point is along the tangent at that point in the direction of motion.
- The velocity vectors v and $\mathbf{v}'$ are as shown in Fig.(a_1).
- ∆v is obtained in Fig. (a_2) using the triangle law of vector addition.
- Since the path is circular, v is perpendicular to r and so is $\mathbf{v}'$ to $\mathbf{r}'$.
- Therefore, ∆v is perpendicular to ∆r. Since average acceleration is along ∆v $\left(\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t}\right)$.

- The average acceleration a is perpendicular to Δr.
- If we place Δv on the line that bisects the angle between r and r'
- We see that it is directed towards the centre of the circle.
- Figure(b) shows them same quantities for smaller time interval.
- Δv and hence a is again directed towards the centre.
- In Fig. (c), $\Delta t \rightarrow 0$ and the average acceleration becomes the instantaneous acceleration. It is directed towards the centre
- Thus, we find that the acceleration of an object in uniform circular motion is always directed towards the centre of the circle.
- Let us now find the magnitude of the acceleration.
- The magnitude of a is, by definition, given by

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{v}|}{\Delta t}$$

- Let the angle between position vectors r and r' be $\Delta\theta$.
- Since the velocity vectors v and v' are always perpendicular to the position vectors, the angle between them is also $\Delta\theta$.
- Therefore, the triangle CPP' formed by the position vectors and the triangle GHI formed by the velocity vectors v, v' and Δv are similar (Fig. 4.19a).
- Therefore, the ratio of the base-length to side-length for one of the triangles is equal to that of the other triangle. That is :

$$\frac{|\Delta \mathbf{v}|}{v} = \frac{|\Delta \mathbf{r}|}{R}$$

$$|\Delta \mathbf{v}| = v\frac{|\Delta \mathbf{r}|}{R}$$

Therefore,

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{v|\Delta \mathbf{r}|}{R\Delta t} = \frac{v}{R} \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t}$$

- If Δt is small, $\Delta\theta$ will also be small and then arc PP' can be approximately taken to be$|\Delta$r$|$:

$$|\Delta \mathbf{r}| \cong v\Delta t$$

$$\frac{|\Delta \mathbf{r}|}{\Delta t} \cong v$$

Or, $$\lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t} = v$$

- Therefore, the centripetal acceleration *ac* is :

$$a_c = \left(\frac{v}{R}\right) v = v^2/R$$

- Thus, the acceleration of an object moving with speed *v* in a circle of radius *R* has a magnitude

v^2/R and is always directed towards the centre.

- This is why this acceleration is called **centripetal acceleration** (a term proposed by Newton).
- "Centripetal" comes from a Greek term which means centre-seeking'.
- Since *v* and *R* are constant, the magnitude of the centripetal acceleration is also constant.
- However, the direction changes — pointing always towards the centre.
- Therefore, a centripetal acceleration is not a **constant vector.**
- We can express centripetal acceleration *ac* in terms of angular speed :
- The time taken by an object to make one revolution is known as its **time period *T*** and
- The number of revolution made in one second is called its **frequency n (=1/*T*)**.
- However, during this time the distance moved by the object is $s = 2\pi r$.
- Therefore, $v = 2\pi r/T = 2\pi\pi n$ In terms of frequency n,
- We have

  W = 2πn $v = 2\pi r n$

  $\mathbf{A_c = 4\pi^2 n^2} \boldsymbol{R}$

THANK YOU

- **BIBLIOGRAPHY**

- **NCERT TEXT BOOK- CLASS XII-PHYSICS**

- **H.P.Sharma**

- **PGT(SS)**
  AECS, Narora.