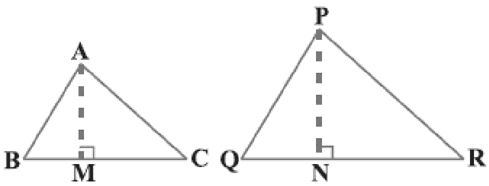
TRIANGLES – MODULE 4/5

Areas of Similar Triangles:

The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Proof:



Consider two triangles, ΔABC and ΔPQR

Given: $\Delta ABC \sim \Delta PQR$

To prove:

$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2$

Construction: Draw altitudes AM and PN of the triangles ABC and PQR from vertices A and P respectively

$ar(\Delta ABC) = \frac{1}{2} \times BC \times AM$

$ar(\Delta PQR) = \frac{1}{2} \times QR \times PN$

$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{\frac{1}{2}\times BC\times AM}{\frac{1}{2}\times QR\times PN} = \frac{BC\times AM}{QR\times PN}$ ------------ (1)

Since $\Delta ABC \sim \Delta PQR$, their corresponding angles are equal. $\angle B = \angle Q$

Also, $\angle M = \angle N = 90°$

Therefore $\Delta ABM \sim \Delta PQN$ by AA similarity criterion

$\frac{AM}{PN} = \frac{AB}{PQ}$ ------------ (2)

Since $\Delta ABC \sim \Delta PQR$, their corresponding sides are proportional

$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ ------------ (3)

From (1) and (3)

$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{AB}{PQ} \times \frac{AM}{PN}$

From (2)

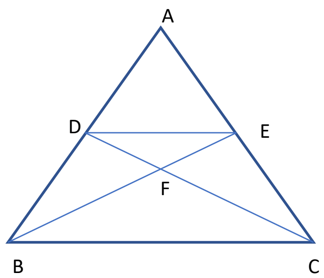
$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \frac{AB}{PQ} \text{x} \frac{AB}{PQ} = \left(\frac{AB}{PQ}\right)^2$

From (3), we get

$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2$$

Solved example

In the given figure, DE || BC and AD:DB = 5:4, find $\frac{ar(\Delta DFE)}{ar(\Delta CFB)}$



Solution:

Consider ΔADE and ΔABC
$\angle A$ is common
$\angle ADE = \angle ABC$ (Since DE || BC and corresponding angles are equal)
$\Rightarrow \Delta ADE \sim \Delta ABC$ (By AA similarity criterion)
$\Rightarrow \frac{AD}{AB} = \frac{DE}{BC}$ (Corresponding parts of similar triangles)-------- (1)

Consider ΔDFE and ΔCFB
$\angle EDF = \angle FCB$ and $\angle DEF = \angle FBC$ (Since DE || BC and alternate interior angles are equal)
$\Rightarrow \Delta DFE \sim \Delta CFB$ (By AA similarity criterion)

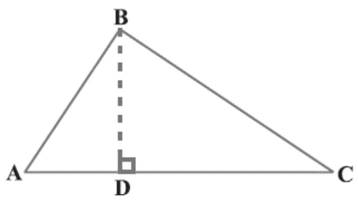
Therefore, the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides
From (1), $\frac{ar(\Delta DFE)}{ar(\Delta \text{CFB})} = \left(\frac{DE}{BC}\right)^2 = \left(\frac{AD}{AB}\right)^2$ -------- (2)

$\frac{AD}{DB} = \frac{5}{4} \Rightarrow \frac{DB}{AD} = \frac{4}{5}$

$\Rightarrow \frac{DB}{AD} + 1 = \frac{DB+AD}{AD} = \frac{AB}{AD} \Rightarrow \frac{AD}{AB} = \frac{1}{\frac{4}{5}+1} = \frac{5}{9}$
From (2)
$\Rightarrow \frac{ar(\Delta DFE)}{ar(\Delta \text{CFB})} = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$

Similarity of triangles in a right triangle:

If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse then triangles on both sides of the perpendicular are similar to the whole triangle and to each other.



Given: ΔABC is a right triangle, $\angle ABC = 90°$

To prove:
$\Delta ADB \sim \Delta ABC$, $\Delta ABC \sim \Delta BDC$ and $\Delta ADB \sim \Delta BDC$

Construction: Draw BD perpendicular to hypotenuse AC

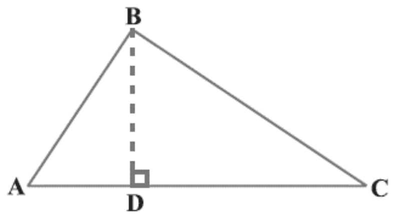
Consider ΔABC and ΔADB
$\angle A$ is common and $\angle ADB = \angle ABC = 90°$
Therefore $\Delta \boldsymbol{ABC} \sim \Delta \boldsymbol{ADB}$ by AA similarity criterion

Consider ΔABC and ΔBDC
$\angle C$ is common and $\angle BDC = \angle ABC = 90°$
Therefore $\Delta \boldsymbol{ABC} \sim \Delta \boldsymbol{BDC}$ by AA similarity criterion

If one triangle is similar to another triangle and this second triangle is similar to a third triangle, then the first triangle is similar to the third triangle
$\Longrightarrow \Delta \boldsymbol{ADB} \sim \Delta \boldsymbol{BDC}$

Pythagoras Theorem:

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



Proof:

Given: ΔABC is a right triangle, $\angle ABC = 90°$

To prove: $AC^2 = AB^2 + BC^2$

Construction: Draw BD perpendicular to hypotenuse AC

Since $\Delta ADB \sim \Delta ABC$, their sides are proportional

$$\frac{AD}{AB} = \frac{AB}{AC}$$

By cross multiplication, $AD \times AC = AB^2$ ---------- (1)

Since $\Delta BDC \sim \Delta ABC$, their sides are proportional

$$\frac{CD}{BC} = \frac{BC}{AC}$$

By cross multiplication, $CD \times AC = BC^2$ ---------- (2)

Adding (1) and (2)

$$AD \times AC + CD \times AC = AB^2 + BC^2$$
$$AC(AD + CD) = AB^2 + BC^2$$
$$AC \times AC = AB^2 + BC^2$$
$$AC^2 = AB^2 + BC^2$$